

Local systems and integrable connections

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1. Local systems

DEFINITION 1.1 (Local system). Let X be a topological space. A sheaf E of complex vector spaces on X is called a *local system* (of complex vector spaces), if X admits an open covering $(U_i)_{i \in I}$, such that $E|_{U_i}$ is isomorphic to the constant sheaf $\mathbf{C}_{U_i}^n$ ($n \in \mathbf{N}$).

In other words, local system is just another name for a locally constant sheaf valued in some complex vector space. A local system E is said to be *trivial*, if it is a *constant* sheaf of complex vector spaces.

REMARK 1.2 (How do local systems arise). In these notes there are two sources of local systems:

- ◇ those from cohomologies of locally topologically trivial fibrations, and
- ◇ those from linear partial differential equations.

The precise content of the first point is: if $f: X \rightarrow B$ is a continuous map which is locally a topologically trivial fibration, then (assuming B is locally contractible) the sheaf $R^i f_* \mathbf{C}_X$ associated with the presheaf $U \mapsto H^i(f^{-1}(U), \mathbf{C})$ is a local system on B .

To explain the second point, let X be an open subset of \mathbf{C} , let $A(t)$ be an $(r \times r)$ -matrix of holomorphic functions on X . Form the system ordinary differential equations

$$\frac{\partial \mathbf{x}}{\partial t} + A(t)\mathbf{x}(t) = 0, \quad \mathbf{x}(t) \in \mathcal{O}(X)^{\oplus r}.$$

If $U \subset X$ is open, define $E(U)$ to be the vector space of holomorphic solutions of this system.

LEMMA 1.3. $E(U)$ is a local system on X .

PROOF. For if $\Delta \subset U$ is a sufficiently small disk, a theorem of Cauchy (see Theorem 4.1) states that the system admits r linearly independent holomorphic solutions over Δ (a fancier version of this is Theorem 2.6). The restriction of these solutions to any smaller open subset of Δ gives a fundamental system of solutions of the system over that smaller open. Whence $U \mapsto E(U)$ is locally constant. \square

EXAMPLE 1.4. Consider the map

$$\begin{array}{ccc} X & \xlongequal{\quad} & \mathbf{C}^2 - \{(x, y) : y^2 = x^3\} \\ f \downarrow & & \\ B & \xlongequal{\quad} & \mathbf{C} - \{0\}. \end{array}$$

defined by $f(x, y) = y^2 - x^3$. For $t \in \mathbf{C}$, let $X(t) = \{(x, y) : y^2 = x^3 + t\}$.

Some observations about the family $\{X(t)\}_{t \in \mathbf{C}^*}$ are in order.

(i) For whatever $t \neq 0$, $X(1)$ is biholomorphic to $X(t)$, an isomorphism is given by

$$\kappa: X(t) \ni (x, y) \mapsto (t^{-1/3}x, t^{-1/2}y) \in X(1).$$

Note that the definition of κ requires a choice of a third root of t , and a square root of t ; thus there are in total 6 such isomorphisms. Indeed, this shows that the family $\{X(t)\}_{t \in \mathbf{C}^*}$ is *analytically locally trivial*: on any simply connected open subset U of \mathbf{C}^* , we can choose uniform branches of the functions $t^{1/3}$ and $t^{1/2}$, and the map

$$X(1) \times U \rightarrow X \times_{\mathbf{C}^*} U = f^{-1}(U), \quad ((x, y), t) \mapsto (t^{-1/3}x, t^{-1/2}y, t)$$

furnishes a fiber-preserving isomorphism of complex manifolds over U .

This isomorphism can only be made over open subsets on which $t^{1/2}$ and $t^{1/3}$ are uniform; thus it is not unreasonable to speculate that the family $\{X(t)\}_{t \in \mathbf{C}^*}$ can not be trivialized globally. To see this we need some information about the topology of $X(t)$.

(ii) The projective completion of $X(t)$ is

$$\bar{X}(t) = \{[x : y : z] \in \mathbf{P}_{\mathbf{C}}^{2, \text{an}} : x^3 + tz^3 - y^2z = 0\}$$

which intersects with the line at infinity defined by $z = 0$ at a single point $[0 : 1 : 0]$. In particular, $X(t)$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$ with a point removed.

(iii) The fibration $f: X \rightarrow B$ is *not* topologically trivial. This can be seen for example by calculating the cohomology of X . Were it trivial, the Künneth formula would imply its first Betti number is three. On the other hand, X is an open subset of $\mathbf{C}^2 - \{0\}$, from the exact sequence

$$H^1(\mathbf{C}^2 - \{0\}) \rightarrow H^1(X) \rightarrow H_{X(0) - \{0\}}^2(\mathbf{C}^2 - \{0\}) \rightarrow H^2(\mathbf{C}^2 - \{0\})$$

and the Thom isomorphism theorem implies that $\dim H^1(X) = 1$, a contradiction.

The first de Rham cohomology group $H^1(X(t))$, as t varies, form a local system on B of rank 2. In view of the previous item, this local system is not trivial.

Next, we explain the distinction between a local system and a general sheaf from the point of view of their étale spaces (Fr: *espaces étalés*). This will allow us to relate local systems with representations of the fundamental group.

DEFINITION 1.5. A topological space $\pi: E \rightarrow X$ over X is said to be an *étale space* over X if π is a local homeomorphism.

A typical separated étale space over X is a covering space with a closed subset removed. In general, an étale space over X is not a separated space over X , for instance the natural projection from the affine line with a doubled origin to the ordinary affine line is not separated.

DEFINITION 1.6 (Étale space of a presheaf). If \mathcal{F} is a presheaf of sets on a topological space X , we can form its *associated étale space* $\pi: E_{\mathcal{F}} \rightarrow X$ as follows:

- ◇ the ambient set of $E_{\mathcal{F}}$ is the disjoint union $\coprod_{x \in X} \mathcal{F}_x$ (recall that elements in the stalk \mathcal{F}_x are called *germs*);
- ◇ for each open subset U of X and $s \in \mathcal{F}(U)$, define $G_{[U, s]}$ to be the collection of germs s_x when x runs through points of U .

The sets $G_{[U, s]}$ is a basis for a topology of $E_{\mathcal{F}}$. $E_{\mathcal{F}}$ being equipped with this topology, the natural projection $E_{\mathcal{F}} \rightarrow X$, which sends a germ to the point it underlies, is an étale space over X ; for if $s \in \mathcal{F}(U)$, the map $x \mapsto s_x$ is a local inverse of $\pi|_{G_{[U, s]}}$.

PROPOSITION 1.7. *The functor*

$$\mathcal{F} \mapsto \left[\begin{array}{c} E_{\mathcal{F}} \\ \downarrow \\ X \end{array} \right]$$

is an equivalence between the category of sheaves on X and the category of étale space over X . A quasi-inverse is given by

$$\left[\begin{array}{c} E \\ \downarrow \pi \\ X \end{array} \right] \mapsto \left[U \mapsto \{ \text{Continuous sections of } \pi \text{ over } U \} \right].$$

(See Godement 1958, II, Théorème 1.2.1.)

Then we have the following.

LEMMA 1.8. *A sheaf \mathcal{F} (of sets) on X is locally constant if and only if $\pi: E_{\mathcal{F}} \rightarrow X$ is a covering space.*

By Lemma 1.8, the étale space of any locally constant sheaf over a simply connected topological space X is isomorphic to $F \times X$ for some discrete space F . Consequently the sheaf is the constant sheaf associated with F .

Now suppose X admits a simply connected covering space $u: \tilde{X} \rightarrow X$. Let F be a right $G = \text{Aut}_X(\tilde{X})$ -set. Then the quotient space of the equivalence relation

$$(\phi, g(x)) \sim (\phi \cdot g, x), \forall g \in G,$$

denoted by $F \times^G \tilde{X}$, is still naturally a covering space over X ; its sheaf of continuous sections is then a locally constant sheaf valued in F . Moreover, every locally constant sheaf on X valued in F comes in this fashion. If $F = \mathbf{C}^n$, and G acts on F by linear transformations, then the resulting locally constant sheaf is a local system of rank n ; conversely any local system on X is from some right G -set F .

When X is a pointed deloopable space (Fr: *espace délaçable*, i.e., a path connected, locally path connected, topological space, such that every point x has a neighborhood U with the property that $\pi_1(U, x) \rightarrow \pi_1(X, x)$ is the trivial map, see Bourbaki 2016), $G^{\text{op}} \simeq \pi_1(X, x)$. The above discussion gives an equivalence between the category of local systems and the category of finite dimensional linear representations of $\pi_1(X, x)$.

2. Connections

Let X be a complex manifold. Let M be an \mathcal{O}_X -Module. As Deligne said in his influential 1970 book *Équations différentielles à points singuliers réguliers*, we can define the notion of a connection on M as follows:

... the ancients would define a (holomorphic) connection on M as the data which assigns to each pair of “infinitesimally close points” (x, y) , an isomorphism $\gamma_{y,x}: M_x \rightarrow M_y$, such that this isomorphism depends in a holomorphic fashion on (x, y) and satisfies $\gamma_{x,x} = \text{Id}$.

Interpreted suitably, this “definition” coincides with the definition given below. But the meaning of “points” should be tweaked: it should mean “points valued in an analytic space” ...

(Deligne 1970, I (2.2))

We will however use a less intuitive definition, due to J.-L. Koszul, using “covariant derivatives”. Although we shall mostly be interested in those M that are coherent over \mathcal{O}_X , the following definition nevertheless works in the greater generality.

DEFINITION 2.1 (Connection on an \mathcal{O}_X -Module). A (holomorphic) *connection* on an \mathcal{O}_X -Module M is a \mathbf{C} -linear map

$$\nabla: M \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} M,$$

such that for any (locally defined) holomorphic function h and any section s of M , we have

$$\nabla(h \cdot s) = dh \otimes s + h \cdot \nabla(s).$$

If ξ is a locally defined tangent vector on M , the notation $\nabla_{\xi}(s)$ signifies the section $\langle \xi, \nabla(s) \rangle$ of M , where $\langle \cdot, \cdot \rangle$ is the pairing between the tangent and cotangent bundle of X .

If M is a *finite locally free* \mathcal{O}_X -Module, it possesses a local frame e_1, \dots, e_r . Then locally on X there exists an $(r \times r)$ matrix of 1-forms $\omega = (\omega_{ij})$ such that

$$\nabla(e_1, \dots, e_r) = (e_1, \dots, e_r)\omega,$$

The matrix ω is called the *connection matrix* with respect to the frame (e_1, \dots, e_r) .

DEFINITION 2.2 (Horizontal morphism). Let M and N be two \mathcal{O}_X -Modules equipped with connections ∇^M and ∇^N , respectively. An \mathcal{O}_X -linear morphism $\varphi: M \rightarrow N$ is said to be *compatible with connections*, or *horizontal*, if

$$\nabla^N \circ \varphi = (\text{Id} \otimes \varphi) \circ \nabla^M.$$

A section s of M is said to be *horizontal* if $\nabla(s) = 0$. Giving a horizontal section of M is equivalent to giving a horizontal morphism from (\mathcal{O}_X, d) into M .

DEFINITION 2.3 (Integrability). A connection $\nabla: M \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} M$ on a \mathcal{O}_X -Module M is said to be *integrable* if it satisfies

$$\nabla_{[\xi, \eta]} = [\nabla_{\xi}, \nabla_{\eta}],$$

for any locally defined vector fields on X . Here, $[\cdot, \cdot]$ is the Lie bracket of operators.

Equivalently, ∇ is integrable if $\nabla^1 \circ \nabla = 0$, where $\nabla^1: \Omega_X^1 \otimes_{\mathcal{O}_X} M \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} M$ is the natural extension of ∇ by “forcing the Leibniz rule”.

Assume M is a finite locally free \mathcal{O}_X -Module. Let ω be the matrix of connection 1-forms with respect to a frame of M . Then ∇ is integrable if and only if $d\omega + \omega \wedge \omega = 0$. The notation “ $d\omega + \omega \wedge \omega$ ” means the following: the operation d applies the exterior derivative component-wise to the matrix-valued 1-form ω , and $\omega \wedge \omega$ denotes matrix multiplication, where the components are wedged rather than multiplied.

REMARK 2.4. Sentences like “ (M, ∇) is a finite locally free \mathcal{O}_X -Module equipped with an integrable connection” are too long. We shall simply say “ M is an integrable connection” instead. (Note that when we say so, we tacitly assume M is finite locally free.)

EXAMPLE 2.5 (Trivial connection). The free \mathcal{O}_X -Module $\mathcal{O}_X^{\oplus r}$, equipped with the exterior differential d , is an integrable connection. An integrable connection is said to be *trivializable*, if it is isomorphic to $(\mathcal{O}_X^{\oplus r}, d)$.

In fact, any *integrable* connection is locally trivializable:

THEOREM 2.6. *Let M be an integrable connection on X . Then X admits an open covering $(U_i)_{i \in I}$, such that $M|_{U_i}$ is isomorphic to a finite direct sum of (\mathcal{O}_{U_i}, d) .*

The theorem can be shown by using a complex analytic version of the Frobenius theorem (see Voisin 2002, Proposition 9.11); alternatively it could be proved by reducing to dimension one (at the cost of introducing fancier concepts). We shall explain the second proof in §3.

The theorem has the following corollary.

COROLLARY 2.7. *Let M be an integrable connection on a complex manifold. Then the sheaf $U \mapsto \text{Ker}(\nabla: M(U) \rightarrow M(U))$ is a local system.*

EXAMPLE 2.8. *If $\varphi: M \rightarrow N$ is a horizontal morphism between integrable connections, then $\text{Ker} \varphi$ and $\text{Coker} \varphi$ are integrable connections.*

This assertion is non-trivial: the kernel and cokernel of a plain \mathcal{O}_X -linear morphism between finite locally free sheaves may not be locally free, unless its rank is locally constant.

To prove the assertion, let us verify φ has locally constant rank. The problem being local, we can prove it with X replaced by any of its open subset, which enables us to use Theorem 2.6. For a morphism $\varphi: \mathcal{O}^n \rightarrow \mathcal{O}^m$ to be horizontal, it is necessary and sufficient that it is represented by a matrix with locally constant coefficients, when has locally constant rank.

In the statement, the integrability is irrelevant, since to prove the constancy of the rank, one can always restrict to curves, over which the connections are automatically integrable.

EXAMPLE 2.9. Let M_1 and M_2 be two \mathcal{O}_X -Modules equipped with connections ∇_1 and ∇_2 respectively. Then we can define connections on the sheaves $\mathcal{H}om_{\mathcal{O}_X}(M_1, M_2)$, $M_1 \otimes_{\mathcal{O}_X} M_2$ as follows: tensor connection formula is

$$\nabla_{\xi}(v_1 \otimes v_2) = \nabla_{1,\xi}(v_1) \otimes v_2 + v_1 \otimes \nabla_{2,\xi}(v_2);$$

the connection on the Hom sheaf is

$$(\nabla_w f)(v_1) = \nabla_{2,w}(f(v_1)) - f(\nabla_{1,w}v_1).$$

If M_1 and M_2 are integrable connections, $\mathcal{H}om_{\mathcal{O}_X}(M_1, M_2)$ and $M_1 \otimes_{\mathcal{O}_X} M_2$ are integrable as well.

When M is a finite and locally free \mathcal{O}_X -Module, the connection on $\mathcal{H}om_{\mathcal{O}}(M, \mathcal{O})$ is called the *dual connection*, and is denoted by M^\vee .

EXAMPLE 2.10. Let $f : X \rightarrow Y$ be a morphism of complex manifolds. The *inverse image* $f^*M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$ of an integrable connection M on Y is again an integrable connection on X .

EXAMPLE 2.11. Let E be a local system on X . Define $M = E \otimes_{\mathbf{C}} \mathcal{O}_X$. Let U be an open subset of X on which E is trivial, and let $v = (v_1, \dots, v_r)$ is a basis of E . Then a section of M over U is given by $\sum_{i=1}^r f_i v_i$, where $f_i \in \mathcal{O}_X(U)$. Define

$$\nabla \sum_{i=1}^r f_i v_i = \sum_{i=1}^r df_i \otimes v_i.$$

As the transition functions between the various frames v are locally constant, it follows that ∇ globalizes, thereby giving rise to an integrable connection on M .

REMARK 2.12 (The “trivial” Riemann–Hilbert correspondence). In view of Theorem 2.6 and Example 2.11, we find that the functors

$$[M, \text{an integrable connection}] \mapsto \text{Ker } \nabla,$$

and

$$[E, \text{a local system}] \mapsto (E \otimes_{\mathbf{C}} \mathcal{O}_X, \nabla)$$

are equivalences between the category of integrable connections on X and the category of local systems on X .

3. Proof of Theorem 2.6

We shall present proof of Theorem 2.6 found in Deligne’s book *Équations différentielles à points singuliers réguliers*. In fact, Deligne proved a more general theorem which is valid in the *relative setting*, i.e., a theorem about a *smooth morphism* $f: X \rightarrow S$ between complex manifolds (or complex analytic spaces). When S reduces to a point, we thus get an *absolute* theorem, i.e., a theorem about the complex manifold X .

Stating the relative version of the theorem allows us to prove the theorem by *dévissage*: “unscrewing” the proof into small steps which are not difficult. Formalism aside, the core is still Cauchy’s theorem (with parameters) in dimension one.

We first prepare some necessary terminologies.

DEFINITION 3.1 (Smooth morphism). A morphism $f: X \rightarrow S$ between complex manifolds is called a *smooth morphism*, provided each point x of X possesses a neighborhood U , such that there exists a commutative diagram of holomorphic maps

$$\begin{array}{ccc} U & \xrightarrow{j} & \mathbf{C}^n \times S \\ & \searrow f|_U & \swarrow \text{pr}_2 \\ & & S \end{array}$$

in which j is an open immersion.

DEFINITION 3.2 (Relative cotangent bundle). Let $f: X \rightarrow S$ be a smooth morphism of complex manifolds. Then the relative cotangent bundle $\Omega_{X/S}^1$ is defined by the exact sequence

$$0 \rightarrow f^* \Omega_S^1 \rightarrow \Omega_X^1 \rightarrow \Omega_{X/S}^1 \rightarrow 0.$$

Note that the left arrow is injective, by virtue of smoothness of f .

The composition $\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \rightarrow \Omega_{X/S}^1$ is denoted by $d_{X/S}$, called the *relative exterior differential*. But we will mostly abuse notation and denote it simply by d .

DEFINITION 3.3 (Relative connection). Let $f: X \rightarrow S$ be a smooth morphism of complex manifolds. Let M be a coherent \mathcal{O}_X -Module. A *relative connection* on M is an $f^{-1}\mathcal{O}_S$ -linear map

$$\nabla: M \rightarrow \Omega_{X/S}^1 \otimes_{\mathcal{O}_X} M.$$

satisfying the ‘‘Leibniz rule’’: $\nabla(fs) = f\nabla(s) + ds \otimes s$; except here we use df to mean $d_{X/S}f$, the *relative exterior differential* of f . The relative connection ∇ is said to be *integrable* if $\nabla \circ \nabla = 0$.

DEFINITION 3.4 (Relative local system). A *relative local system* on X is an $f^{-1}\mathcal{O}_S$ -Module, locally isomorphic to $f^{-1}\mathcal{F}$ for some coherent \mathcal{O}_S -Module \mathcal{F} .

Now we can state the relative version of Theorem 2.6.

THEOREM 3.5. *Let $f: X \rightarrow S$ be a smooth morphism between complex manifolds. Let (M, ∇) be a relative integrable connection on X . Then $U \mapsto \text{Ker } \nabla$ is a relative local system on X . (See Deligne 1970, Chapitre I, Théorème 2.23.)*

The proof of Theorem 3.5 occupies the rest of this section.

Initial case: $X = S \times \Delta$, $S = \Delta^n$, $f = \text{pr}_1$, M **free**. Let s_0 be the zero section $S \simeq S \times \{0\} \hookrightarrow X$. Then $\mathcal{F} = s_0^*M$ is a free \mathcal{O}_S -Module on S . For each section v of \mathcal{F} , the existence and uniqueness theorem of Cauchy problem with parameters (whose proof is recalled in §4) ensures that there is a horizontal section \tilde{v} of M such that $v = \tilde{v}|_{S \times \{0\}}$. Let v be a frame of \mathcal{F} . Apply the above to v yields a frame \tilde{v} of M which is horizontal.

Second case: $X = S \times \Delta$, $S = \Delta^n$, $f = \text{pr}_1$. By shrinking S and X to smaller disks, we can assume M admits a presentation

$$M_1 \xrightarrow{d} M_0 \xrightarrow{\epsilon} M \rightarrow 0$$

where M_0 and M_1 are free. By restricting further, we could assume M_1 and M_0 are equipped with connections such that ϵ and d are compatible with the connection. (If e is a basis of M_0 , it suffices to determine $\nabla_0(e)$. One can define it to be any one in the preimage of $\nabla(\epsilon e)$; define ∇_1 similarly.)

Since the relative dimension is one, ∇_0, ∇_1 are automatically integrable. By the initial case, there exists coherent \mathcal{O}_S -Modules \mathcal{F}_0 and \mathcal{F}_1 such that $(M_i, \nabla_i) \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{F}_i$. It follows that $M \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{F}_0/d\mathcal{F}_1$.

Third case: f is of relative dimension 1. This follows immediately from the second case as S is locally isomorphic to a polydisk. We remark that this also works if S is singular, i.e., an arbitrary complex analytic space. This is because locally S can be embedded into a polydisk Δ^n and coherent sheaves on S are locally restricted from Δ^n .

General case. Proceed by induction on the relative dimension n . The base case ($n = 0$) is trivial, we assume $n > 0$.

The problem being local, we can assume $X \cong S \times D^n = S \times D^{n-1} \times D$. Set $X_0 = S \times D^{n-1} \times \{0\}$, and $p: X \rightarrow X_0$ be the projection to the first two factors.

The restriction of M to X_0 is a relative integrable connection M_0 on X_0 . By inductive hypothesis,

$$(3.6) \quad M_0 \cong \mathcal{O}_{X_0} \otimes_{\text{pr}_1^{-1}\mathcal{O}_S} \mathcal{F}$$

for some coherent sheaf \mathcal{F} on S .

On the other hand, (M, ∇) also defines an integrable connection relative to $S \times D^{n-1} \simeq X_0$. Thus there exists a coherent sheaf \mathcal{G} on X_0 such that

$$(3.7) \quad M \cong \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_{X_0}} \mathcal{G}.$$

As we have seen in the first and second case, \mathcal{G} is isomorphic to M_0 . Combining (3.6) and (3.7) gives an isomorphism $\alpha: M \xrightarrow{\sim} \mathcal{O}_X \otimes_{\text{pr}_1^{-1}\mathcal{O}_S} \mathcal{F}$. It satisfies:

- ◊ the restriction of α to X_0 is horizontal;

◇ α is “relatively horizontal” for p .

From these two points we can deduce the horizontality of α , as follows.

Let v be a section of \mathcal{F} . Then v can be viewed as a horizontal section of M_0 . The second point implies that $\nabla_{\partial_n} v = 0$. It follows from the integrability that

$$\nabla_{\partial_n} \nabla_{\partial_i} v = \nabla_{\partial_i} \nabla_{\partial_n} v = 0.$$

Thus $\nabla_{\partial_i} v$ are horizontal relative to p . By the first point, however, $\nabla_{\partial_i} v = 0$ on X_0 . Therefore $\nabla v = 0$, meaning v is indeed horizontal for the relative connection M . This proves that α is horizontal and finishes the proof of Theorem 3.5.

4. Cauchy’s theorem with parameters

The reference for this section is Forster 1981, §11.

Fix a positive real number R (could be $+\infty$). Let $\Delta = \{t \in \mathbf{C} : |t| < R\}$. Let S be a complex manifold (or a complex analytic space). Let $A(s, t)$ be an $(n \times n)$ -matrix whose entries are holomorphic functions on $S \times \Delta$.

Consider the following family of systems of ordinary differential equations parametrized by $s \in S$:

$$(*) \quad \frac{\partial \mathbf{x}}{\partial t} - A(s, t)\mathbf{x}(s, t) = 0, \quad \mathbf{x}(s, t) \in \mathcal{O}(S \times \Delta)^{\oplus n}.$$

THEOREM 4.1. *Fix $\mathbf{x}_0 \in \mathcal{O}(S \times \{0\})^n$. There exists a unique $\mathbf{x} \in \mathcal{O}(S \times \Delta)^{\oplus n}$ satisfying $(*)$ and $\mathbf{x}(s, 0) = \mathbf{x}_0(s)$.*

PROOF OF UNIQUENESS. Write

$$A(t) = \sum_{\nu=0}^{\infty} A_{\nu}(s)t^{\nu}, \quad A_{\nu} \in M_n(\mathcal{O}(S)).$$

Let $\mathbf{x}(t) = \sum_{\nu=0}^{\infty} c_{\nu}(s)t^{\nu}$ be a solution of $(*)$. Then

$$\sum_{k=1}^{\infty} k c_k t^{k-1} = \sum_{k=0}^{\infty} \left(\sum_{\mu+\nu=k} A_{\mu} c_{\nu} \right) t^k.$$

Thus

$$c_{k+1} = \frac{1}{k+1} \sum_{\nu=0}^k A_{k-\nu} c_{\nu}, \quad k = 0, 1, \dots$$

Once $c_0(s) = \mathbf{x}(s, 0)$ is determined, the above recursive equation determines all $c_k(s)$, $k \in \mathbf{N}$. This proves the uniqueness. \square

PROOF OF EXISTENCE. Keep the above notation above. Fix any compact polydisk P of S , and a real number $0 < r < R$.

Since the entries of $A(s, t)$ are analytic on $P \times \{|t| < r\}$, there exists $N \in \mathbf{N}$ such that

$$\|A_{\nu}(s)\|_{P, \infty} = \max\{|a_{ij,\nu}(s)| : 1 \leq i, j \leq n; s \in P\} \leq N r^{-\nu-1},$$

where $a_{ij,\nu}(s)$ are the entries of A_{ν} .

Define

$$b_{ij}(t) = \frac{N}{r} \left(1 - \frac{t}{r}\right) = \frac{N}{r} \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{r^{\nu}}.$$

Consider an auxiliary equation (independent of s)

$$\mathbf{y}' = B(t)\mathbf{y},$$

with initial condition $\mathbf{y}(0) = (K, \dots, K)^T$. Here, K is any real number which is larger than $\|\mathbf{x}(s)_0\|_{P, \infty} < \infty$.

The auxiliary equation is easily solved by $\mathbf{y}(t) = (\psi(t), \dots, \psi(t))^T$, where

$$\psi(t) = K \left(1 - \frac{t}{r}\right)^{-nN}.$$

The solution $\mathbf{y}(t)$ is visibly convergent on $\{|t| < r\}$.

If we write

$$\mathbf{y}(t) = \sum_{\nu=0}^{\infty} \gamma_{\nu} t^{\nu},$$

then by the uniqueness step the vectors γ_{ν} and the matrices B_{ν} satisfy the recursive relation

$$\gamma_{k+1} = \frac{1}{k+1} \sum_{\nu=0}^k B_{k-\nu} \gamma_{\nu}.$$

By induction and the hypothesis on the matrix B , we conclude that $\mathbf{y}(t)$ is a majorant of the series $\mathbf{x}(s, t)$ on $P \times \{|t| < r\}$, i.e.,

$$\|c_k(s)\|_{P, \infty} < \gamma_k.$$

This forces \mathbf{x} to be convergent in $P \times \{|t| < r\}$. Since r is arbitrary, $\mathbf{x}(t)$ converges on $P \times \Delta$. Since P is arbitrary, the uniqueness implies \mathbf{x} exists globally on $S \times \Delta$. \square

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