Deligne's Lefschetz theorem for perverse sheaves

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In this note we present a proof of Deligne's weak Lefschetz theorem for perverse sheaves, recorded in N. Katz (1993), "Affine cohomological transforms, perversity, and monodromy", Appendix. See Theorem 3.3. Throughout this note, we fix an algebraically closed field k. Let ℓ be a prime number different from the characteristic of k.

1. Perversity

For an algebraic variety X over k, denote by $\mathrm{D}_{\mathrm{X}} = \mathrm{D}^b_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$ the derived category of complexes of $\overline{\mathbf{Q}}_\ell$-sheaves on X with constructible cohomology, and by ${}^{\mathfrak{p}}\mathrm{D}^{\leq 0}_{\mathrm{X}}$ the full subcategory of D_{X} consisting of constructible complexes satisfying the *support condition*, i.e., objects K of D_{X} satisfying

$$(*)\qquad \dim \operatorname{Supp} \mathcal{H}^i(\mathrm{K}) \leq -i, \quad \text{for all } i \in \mathbf{Z}.$$

We recall that if $\mathcal{F}$ is a sheaf on X, its support $\operatorname{Supp}\mathcal{F}$ is defined as the *closure* of points $x \in \mathrm{X}$ such that $\iota_x^*\mathcal{F} \neq 0$, where $\iota_x \colon \{x\} \to \mathrm{X}$ is the inclusion map.

Let ${}^{\mathfrak{p}}\mathrm{D}^{\geq 0}_{\mathrm{X}}$ be the full subcategory of D_{X} consisting of objects K such that $\mathbf{D}\mathrm{K}$ is an object of ${}^{\mathfrak{p}}\mathrm{D}^{\leq 0}_{\mathrm{X}}$, where $\mathbf{D}\mathrm{K}$ is the Verdier dual of K. The pair $({}^{\mathfrak{p}}\mathrm{D}^{\leq 0}_{\mathrm{X}}, {}^{\mathfrak{p}}\mathrm{D}^{\geq 0}_{\mathrm{X}})$ form a t-structure of D_{X} known as the *perverse t-structure*. Its *heart*, the abelian category ${}^{\mathfrak{p}}\mathrm{D}^{\geq 0}_{\mathrm{X}} \cap {}^{\mathfrak{p}}\mathrm{D}^{\leq 0}_{\mathrm{X}}$, is known as the category of *perverse sheaves*.

Example 1.1. Let X be an algebraic variety of dimension $\leq n$. Let $\mathcal{F}$ be any constructible sheaf on X. Then $\mathcal{F}[n]$ satisfies the support condition. Indeed, for $i \neq -n$ we have $\mathcal{H}^i(\mathcal{F}[n]) = 0$, so $(*)$ is automatically verified; for $i = -n$, we have $\operatorname{Supp}\mathcal{H}^{-n}(\mathcal{F}[n]) = \operatorname{Supp}\mathcal{F}$, which has dimension $\leq n$ since $\dim \mathrm{X} \leq n$.

Example 1.2. Let X be a *smooth* variety over k of pure dimension n. In particular, if $\mathcal{L}$ is a lisse $\overline{\mathbf{Q}}_\ell$-sheaf, $\mathbf{D}\mathcal{L} = \mathcal{L}^*[-2n]$, where $\mathcal{L}^*$ is the $\overline{\mathbf{Q}}_\ell$-dual of $\mathcal{L}$. By Example 1.1, $\mathcal{L}[n]$ satisfies the support condition. It follows that $\mathcal{L}^*[n] = \mathbf{D}(\mathcal{L}[n])$ satisfies the cosupport condition. Therefore, for any lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{L}$ on X, $\mathcal{L}[n]$ is a perverse sheaf.

Now assume X is an *affine variety* over k. Recall that the *Artin vanishing theorem* asserts that if $\mathcal{F}$ is a constructible $\overline{\mathbf{Q}}_\ell$-sheaf on an algebraic variety of dimension $\leq n$, then

$$\mathrm{H}^i(\mathrm{X}; \mathcal{F}) = 0 \quad \text{for any } i > n.$$

Now if K is an object of ${}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\leq 0}$, then $\mathcal{H}^i(\mathrm{K})$ are constructible $\overline{\mathbf{Q}}_\ell$-sheaves. Since $\operatorname{Supp}\mathcal{H}^i(\mathrm{K})$ is a closed subset of X, it is affine. The support condition ensures that

$$\mathrm{H}^i(\mathrm{X};\mathcal{H}^j(\mathrm{K})) = 0, \text{ for any } i > -j.$$

There is a spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(\mathrm{X};\mathcal{H}^j(\mathrm{K})) \Rightarrow \mathrm{H}^{i+j}(\mathrm{X};\mathrm{K}).$$

If X is affine, then the above discussion shows that $\mathrm{E}_2^{i,j} = 0$ for any $i + j > 0$. Therefore, we conclude the following.

Theorem 1.3. *Let* X *be an affine variety over* k. *Let* K *be an object of* ${}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\leq 0}$. *Then*

$$\mathrm{H}^i(\mathrm{X};\mathrm{K}) = 0 \textit{ for any } i > 0.$$

Applying Verdier duality (i.e., $\mathrm{H}^i(\mathrm{X};\mathrm{K}) \simeq \mathrm{H}_c^{-i}(\mathrm{X};\mathbf{D}\mathrm{K})^*$ for any $\mathrm{K} \in \mathrm{D}_{\mathrm{X}}$), we obtain:

Theorem 1.4. *Let* X *be an affine variety over* k. *Let* K *be an object of* ${}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\geq 0}$. *Then*

$$\mathrm{H}_c^i(\mathrm{X};\mathrm{K}) = 0 \textit{ for any } i < 0.$$

2. *t*-exactness

Let X and Y be two algebraic varieties over k. Let $\mathrm{F}\colon \mathrm{D}_{\mathrm{X}} \to \mathrm{D}_{\mathrm{Y}}$ be an exact functor. We say F is *right t-exact* (with respect to the perverse t-structure), if $\mathrm{F}({}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\leq 0}) \subset {}^{\mathfrak{p}}\mathrm{D}_{\mathrm{Y}}^{\leq 0}$; and we say F is *left t-exact* if $\mathrm{F}({}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\geq 0}) \subset {}^{\mathfrak{p}}\mathrm{D}_{\mathrm{Y}}^{\geq 0}$. If F is both right t-exact and left t-exact, we shall simply say F is *t-exact*.

Example 2.1. Let $f\colon \mathrm{Y} \to \mathrm{X}$ is a *quasi-finite* morphism. Then $f_!$ is right t-exact, and, by duality, $\mathrm{R}f_*$ is left t-exact. In particular, if f is finite, then $\mathrm{R}f_* = f_* = f_!$ is t-exact.

Indeed, if f is quasi-finite, then $f_!\colon \mathrm{D}_{\mathrm{Y}} \to \mathrm{D}_{\mathrm{X}}$ is exact with respect to the usual t-structure, i.e., $\mathcal{H}^i(f_!\mathrm{K}) = f_!\mathcal{H}^i(\mathrm{K})$ for any object K of D_{Y}. In particular, the support of $\mathcal{H}^i(f_!\mathrm{K})$ is the closure of the $f(\operatorname{Supp}\mathcal{H}^i(\mathrm{K}))$, and thus have the same dimension.

Example 2.2. Let $f\colon \mathrm{Y} \to \mathrm{X}$ be an *étale* morphism. Then $\mathrm{R}f^! = f^*$ is t-exact. Indeed, f^* is exact with respect to the t-structures, thus $\mathcal{H}^i(f^*\mathrm{K}) = f^*(\mathcal{H}^i(\mathrm{K}))$. It follows that the support of $\mathcal{H}^i(f^*\mathrm{K})$ equals $f^{-1}(\operatorname{Supp}(\mathcal{H}^i\mathrm{K}))$.

Example 2.3. A common generalization of Theorem 1.3 and Theorem 1.4 is the following version of Artin's vanishing theorem: if $f\colon \mathrm{X} \to \mathrm{Y}$ is an *affine* morphism, then $\mathrm{R}f_*$ is right t-exact, and $\mathrm{R}f_!$ is left t-exact. See SGA 4, XIV 3.1. Combining with the results in Example 2.1, we find that if $j\colon \mathrm{U} \to \mathrm{X}$ is an open immersion as well as an affine morphism (e.g. the inclusion of the complement of a Cartier divisor), then $j_!$ and $\mathrm{R}j_*$ are t-exact.

Example 2.4. Let $f\colon \mathrm{X} \to \mathbf{A}^1$ be a regular function. Grothendieck introduced the nearby cycle functor $\mathbf{R}\Psi_f$ and vanishing cycle functor $\mathbf{R}\Phi_f$ associated to f. These can be viewed as functors from D_{X} to $\mathrm{D}_{\mathrm{X}_0}$, where $\mathrm{X}_0 = f^{-1}(0)$.

It is a theorem of Gabber that $\mathbf{R}\Psi_f[-1]$ is t-exact, and a theorem of Beilinson that $\mathbf{R}\Phi_f[-1]$ is t-exact.

3. Deligne's theorem

We fix an integer $n \geq 1$, and denote by $\mathbf{P}$ the n-dimensional projective space over k. We denote by $\check{\mathbf{P}}$ the *dual* projective space over k. That is, $\check{\mathbf{P}}$ is the space of hyperplanes in $\mathbf{P}$. Let $\mathcal{H} \subset \mathbf{P} \times \check{\mathbf{P}}$ be the universal hyperplane.

Lemma 3.1. *Let $f\colon \mathrm{X} \to \mathbf{P}$ be a separated morphism of finite type. Let* K *be an object of* D_{X}. *On the product* $\mathrm{X} \times \check{\mathbf{P}}$, *consider the object* $\mathrm{pr}_1^*\mathrm{K}$ *and the cartesian diagram*

$$\begin{array}{ccc} (\mathrm{X}\times\check{\mathbf{P}}) - (f\times\mathrm{Id})^{-1}\mathcal{H} & \xrightarrow{g} & (\mathbf{P}\times\check{\mathbf{P}}) - \mathcal{H} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathrm{X}\times\check{\mathbf{P}} & \xrightarrow{f\times\mathrm{Id}} & \mathbf{P}\times\check{\mathbf{P}} \end{array}.$$

On $\mathbf{P}\times\check{\mathbf{P}}$, *the natural map of adjunction*

$$\beta_! \mathrm{R}g_*(\alpha^*\mathrm{pr}_1^*\mathrm{K}) \to \mathrm{R}(f\times\mathrm{Id})_*\alpha_!(\alpha^*\mathrm{pr}_1^*\mathrm{K})$$

is an isomorphism.

Proof. Let $\mathcal{Y} = (\mathrm{X}\times\check{\mathbf{P}}) \times_{\mathbf{P}\times\check{\mathbf{P}}} \mathcal{H}$. Consider the fiber diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{h} & \mathcal{H} \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle i} \\ \mathrm{X}\times\check{\mathbf{P}} & \xrightarrow{f\times\mathrm{Id}} & \mathbf{P}\times\check{\mathbf{P}} \end{array}.$$

Then $\mathcal{H}$ and $\mathbf{P}\times\check{\mathbf{P}}$ are smooth over $\mathbf{P}$. Since smooth morphism is stable under base change, $\mathcal{Y}$ and $\mathrm{X}\times\check{\mathbf{P}}$ are smooth over X.

By smooth base change, $\mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) = \mathrm{pr}_1^*(\mathrm{R}f_*(\mathrm{K}))$ is induced from a constructible complex on $\mathbf{P}$. Therefore, by the smoothness of $\mathcal{H}$ and $\mathbf{P}\times\check{\mathbf{P}}$ over $\mathbf{P}$, and the relative purity isomorphism, we have

$$i^*\mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) \simeq \mathrm{R}i^!(\mathrm{R}(f\times\mathrm{Id})_*\mathrm{K})[-2]$$

It follows from duality and proper base change that

$$\begin{aligned} i^*\mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) &= \mathrm{R}i^!\mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K})[-2] \\ &= \mathrm{R}h_*\mathrm{R}\iota^!(\mathrm{pr}_1^*\mathrm{K})[-2]. \end{aligned}$$

Now we apply relative purity to the smooth pair $\mathcal{Y}\to\mathrm{X}\times\check{\mathbf{P}}$ over X, and the complex $\mathrm{pr}_1^*\mathrm{K}$. We find that $\mathrm{R}\iota^!(\mathrm{pr}_1^*\mathrm{K})[-2] \simeq \iota^*\mathrm{pr}_1^*\mathrm{K}$. We can now conclude that

$$i^*\mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) \simeq \mathrm{R}h_*\iota^*(\mathrm{pr}_1^*\mathrm{K}).$$

From the above discussion we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{R}(f\times\mathrm{Id})_*(\alpha_!\alpha^*(\mathrm{pr}_1^*\mathrm{K})) & \longrightarrow & \beta_!\beta^*(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) & = & \beta_!\mathrm{R}g_*\alpha^*\mathrm{pr}_1^*\mathrm{K} \\ \downarrow & & \downarrow & & \\ \mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) & = & \mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) & & \\ \downarrow & & \downarrow & & \\ \mathrm{R}(f\times\mathrm{Id})_*(\iota_*\iota^*(\mathrm{pr}_1^*\mathrm{K})) & \longrightarrow & i_*i^*\mathrm{R}(f\times\mathrm{Id})_*(\mathrm{pr}_1^*\mathrm{K}) & & \end{array},$$

in which all two columns are distinguished triangles. Since the horizontal arrow in the third row is an isomorphism by the previous discussion, we conclude that the first horizontal arrow is an isomorphism as well. This completes the proof. □

Applying generic base change to the above universal situation gives the following corollary.

Corollary 3.2. *In the situation of Lemma 3.1, for a hyperplane* L *in* $\mathbf{P}$, *form the cartesian diagram*

$$\begin{array}{ccc} \mathrm{X}-f^{-1}\mathrm{L} & \xrightarrow{g_{\mathrm{L}}} & \mathbf{P}-\mathrm{L} \\ \downarrow{\scriptstyle \alpha_{\mathrm{L}}} & & \downarrow{\scriptstyle \beta_{\mathrm{L}}} \\ \mathrm{X} & \xrightarrow{f} & \mathbf{P} \end{array} .$$

Then there exists a Zariski open dense $\mathcal{U}$ *subset of* $\check{\mathbf{P}}$, *such that for any* $[\mathrm{L}] \in \mathcal{U}$, *the natural map*

$$\beta_{\mathrm{L}!}\mathrm{R}g_{\mathrm{L}*}\alpha_{\mathrm{L}}^{*}\mathrm{K} \to \mathrm{R}f_{*}\alpha_{\mathrm{L}!}\alpha_{\mathrm{L}}^{*}\mathrm{K}$$

is an isomorphism. Taking global section yields an isomorphism

$$\mathrm{H}_c^{*}(\mathbf{P}-\mathrm{L};\mathrm{R}g_{\mathrm{L}*}(\alpha_{\mathrm{L}}^{*}\mathrm{K})) \simeq \mathrm{H}^{*}(\mathrm{X}, f^{-1}\mathrm{L};\mathrm{K}).$$

Now we can state and prove Deligne's version of the weak Lefschetz theorem.

Theorem 3.3. *Let* k *be an algebraically closed field. Let* $f\colon \mathrm{X} \to \mathbf{P}^n$ *be a quasi-finite morphism of separated schemes. Let* K *be an object of* ${}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\geq 0}$ *on* X. *Then for a generic hyperplane* L *in* $\mathbf{P}^n$, *we have*

$$\mathrm{H}^{i}(\mathrm{X}, \mathrm{X}\cap f^{-1}\mathrm{L};\mathrm{K}) = \{0\}$$

for $i < 0$. *Equivalently, the restriction map*

$$\mathrm{H}^{i}(\mathrm{X};\mathrm{K}) \to \mathrm{H}^{i}(\mathrm{X}\cap f^{-1}\mathrm{L};\mathrm{K}|_{\mathrm{X}\cap f^{-1}\mathrm{L}})$$

is injective if $i = -1$, *and bijective if* $i < -1$.

Proof. In view of Corollary 3.2, for L generic $\mathrm{H}^{i}(\mathrm{X}, \mathrm{X}\cap f^{-1}\mathrm{L};\mathrm{K})$ can be identified with $\mathrm{H}_c^{*}(\mathbf{P}-\mathrm{L};\mathrm{R}g_{\mathrm{L}*}\alpha_{\mathrm{L}}^{*}\mathrm{K})$. Since K is an object of ${}^{\mathfrak{p}}\mathrm{D}_{\mathrm{X}}^{\geq 0}$, α_{L}^{*} is t-exact, and $\mathrm{R}g_{\mathrm{L}*}$ is left t-exact (see Example 2.1, we see $\mathrm{R}g_{\mathrm{H}*}\alpha_{\mathrm{H}}^{*}\mathrm{K} \in {}^{\mathfrak{p}}\mathrm{D}_{\mathbf{P}-\mathrm{L}}^{\geq 0}$. The theorem now follows from Theorem 1.4. □

Email address: zhangdingxin13@gmail.com