Verdier duality

Contents

1. Poincaré duality and contravariant Verdier duality

Let M be a topological manifold (without assuming M is second countable). Then the Poincaré duality theorem states that if M is "orientable", then there is a duality isomorphism

$$\mathrm{D_M}\colon \mathrm{H}^k_c(\mathrm{M};\mathbf{Z}) \xrightarrow{\sim} \mathrm{H}_{n-k}(\mathrm{M};\mathbf{Z}). \tag{1.1}$$

See for example Hatcher 2002, Theorem 3.35. However, if M has even one singular point, the above isomorphism breaks down. For example, consider $\mathrm{X} = \mathrm{S}(\mathbf{S}^1 \times \mathbf{S}^1)$, the suspension of a 2-dimensional torus, which is a compact 3-dimension space with has the north and south poles as singular points. Then $\mathrm{H}^2(\mathrm{X}) \simeq \mathrm{H}^1(\mathbf{S}^1 \times \mathbf{S}^1) \simeq \mathbf{Z}^2$, but $\mathrm{H}^1(\mathrm{X}) \simeq \widetilde{\mathrm{H}}^0(\mathbf{S}^1 \times \mathbf{S}^1) = 0$. Since $\mathrm{H}^1(\mathrm{X}) = \mathrm{Hom}(\mathrm{H}_1(\mathrm{X}), \mathbf{Z})$, we find that Poincaré duality fails. (Historically, this is also the example that Poincaré used to illustrate that his duality theorem should only be expected for manifolds.)

A generalization of Poincaré duality to singular spaces was introduced by Borel and Moore 1960. A further generalization to maps between spaces is provided by Verdier 1966, as an analogue of Grothendieck's coherent duality theorem for complexes of coherent sheaves. Let us first explain the "absolute case", essentially due to Borel and Moore, which resembles the most to the classical Poincaré duality, and then explain the "relative case".

Let X be a locally compact topological space with finite covering dimension. Verdier showed that [1] there is an object $\omega_\mathrm{X} \in \mathsf{D}^+(\mathrm{X})$, called the *dualizing complex*, such that for any object $\mathcal{F}$ of $\mathsf{D}^+(\mathrm{X})$, we have

$$\mathrm{RHom}_{\mathsf{D}^+(\mathrm{X})}(\mathcal{F}, \omega_\mathrm{X}) \simeq \mathrm{R}\Gamma_c(\mathrm{X};\mathcal{F})^\vee,$$

where for an object K of $\mathsf{D}^+(\mathbf{Z})$, $\mathrm{K}^\vee = \mathrm{RHom}(\mathrm{K}, \mathbf{Z}_\mathrm{X})$ is its "derived dual". If we write $\mathbf{D}\mathcal{F} = \mathrm{R}\mathscr{H}\!om(\mathcal{F}, \omega_\mathrm{X})$, ("derived sheaf Hom") then the above equation can be rewritten as

$$\mathrm{R}\Gamma(\mathrm{X};\mathbf{D}\mathcal{F}) \simeq \mathrm{R}\Gamma_c(\mathrm{X};\mathcal{F})^\vee. \tag{1.2}$$

Specializing to $\mathcal{F} = \mathbf{Z}_\mathrm{X}$, (1.2) becomes

$$\mathrm{R}\Gamma(\mathrm{X};\omega_\mathrm{X}) \simeq \mathrm{R}\Gamma_c(\mathrm{X};\mathbf{Z})^\vee. \tag{1.3}$$

When X is an orientable manifold of pure dimension n, it can be shown that $\omega_\mathrm{X} \simeq \mathbf{Z}_\mathrm{X}[n]$, and we obtain by using the universal coefficient theorem the following exact sequence:

$$0 \to \mathrm{Ext}^1(\mathrm{H}^{k+1}_c(\mathrm{X};\mathbf{Z}), \mathbf{Z}) \to \mathrm{H}^{n-k}(\mathrm{X};\mathbf{Z}) \to \mathrm{Hom}(\mathrm{H}^k_c(\mathrm{X};\mathbf{Z}), \mathbf{Z}) \to 0, \tag{1.4}$$

which is a variant of (1.1). When using a field, e.g., $\mathbf{R}$, instead of $\mathbf{Z}$, as coefficients, the Ext-group vanishes, and we get $\mathrm{H}^{n-k}(\mathrm{X};\mathbf{R}) \simeq \mathrm{H}^k_c(\mathrm{X};\mathbf{R})^\vee$. This is what is usually proved in textbooks teaching de Rham's theory, such as Bott and Tu 1982, Remark 5.7.

Remark 1.5 (Dualizing complex as the complex of geometric Borel–Moore chains). When the space X admits a finite dimensional PL structure, the dualizing complex ω_X has a relatively explicit chain model. To describe it, we adopt the following rule switching between homological and cohomological conventions: for a chain complex $C_\bullet$, we regard it as a cochain complex with $C^i = C_{-i}$.

For each admissible triangulation K of X, let $C^{BM}_\bullet(K)$ be the infinite formal linear combinations of simplices of K. If $\xi = \sum_{\sigma\in K_k} n_\sigma \sigma$ is a simplicial k-chain, its *support* $\mathrm{Supp}(\xi)$ is the union of the simplices σ such that $n_\sigma \neq 0$. If K' is a refinement of K, there is a natural map $r\colon C^{BM}(K) \to C^{BM}(K')$. Moreover, $\mathrm{Supp}(\xi) = \mathrm{Supp}(r(\xi))$. We define $C^{BM}_\bullet(X)$ as $\mathrm{colim}_K\, C^{BM}_\bullet(K)$, where K runs through the partially ordered set of all admissible triangulations of X. An element of $C^{BM}_\bullet(X)$ is called a *geometric Borel–Moore chain*; any geometric chain ξ has a well-defined support $\mathrm{Supp}(\xi)$, and it is a locally closed subset of X. Then we can describe $\omega_X(U)$ as the subcomplex $C^{BM}_\bullet(X)$ comprised by the Borel–Moore chains whose supports are contained in U.

We shall prove the concrete description that $\omega_X(U) \simeq C^{BM}(U)$ in Remark 3.11.

Remark 1.6. It is clear that (1.1) ⇒ (1.4). One of course cannot deduce (1.1) from (1.4), nor from (1.3) directly, even if we use the field coefficients. Since taking double dual is not an isomorphism for infinite dimensional spaces. Nevertheless, if X is a manifold, both $\mathbf{Z}_X$ and ω_X are "perfect complexes", i.e., locally isomorphic to bounded complexes whose entries are finitely generated locally constant sheaves, so the local version of (1.1) and (1.3) are equivalent. One can then use Mayer–Vietoris to globalize, just like the usual proof of Poincaré duality.

Now we explain the relative version of the Verdier duality. Let $f\colon X \to Y$ be a continuous map between locally compact topological spaces of finite covering dimension. The relative version of the compactly supported cohomology is the direct image with proper support $f_!$. In this context, the Verdier duality theorem asserts that $f_!\colon D^+(X) \to D^+(Y)$ *has a right adjoint* $f^!\colon D^+(Y) \to D^+(X)$. Therefore, we have isomorphisms

$$\mathrm{RHom}_{D^+(X)}(\mathcal{F}, f^!\mathcal{G}) \simeq \mathrm{RHom}_{D^+(Y)}(f_!\mathcal{F}, \mathcal{G}) \tag{1.7}$$

bifunctorial with respect to $\mathcal{F}$ and $\mathcal{G}$. The absolute Verdier duality is obtained by setting Y to be a point, $\mathcal{G} = \mathbf{Z}$, and $\omega_X = f^!\mathbf{Z}$.

For a direct proof of (1.7), the reader is referred to Verdier's original Bourbaki report, or consult textbooks like Kashiwara and Schapira 1990, Chapter III or Gelfand and Manin 2003, Theorem III.8.16.

In this lecture, we will deviate slightly from the above classical narrative, and present a variant of the Verdier duality, called the *covariant Verdier duality*, using the language of cosheaves. See Section 3. This variant is aligned more with the "homological" formulation (1.1) of the Poincaré duality rather than the "cohomological" one (1.4). This covariant version not only implies the classical formulation (1.7) (see Remark 3.6), but also overcomes the inconvenience mentioned in Remark 1.6.

2. Sheaves and cosheaves

In this section, we shall revisit the concepts of sheaves and cosheaves within the context of ∞-categories. (For an introduction to the language of ∞-categories, visit Kerodon.) By adopting this broader perspective, the proof of covariant Verdier duality (Theorem 4.5 below) becomes somewhat more streamlined. Nevertheless, our intention is not to propose a complete departure from the traditional conventions and principles of sheaf theory. Instead, this extended framework should be seen as a helpful tool that complements the existing theory rather than supplants it.

[2] Definition 2.1. Let $\mathcal{C}$ be an ∞-category. Let X be a topological space. In the derived sense, a $\mathcal{C}$-valued *sheaf* on X is a *contravariant* functor $\mathcal{F}$ from the partially ordered set of open subsets of X into $\mathcal{C}$, such that for any open U of X, any open covering $(U_\alpha)_{\alpha\in A}$ of U, the limit $\lim_V \mathcal{F}(V)$ exists—here V ranges over all open subsets of U which are contained in some U_α—and the canonical map

$$\mathcal{F}(U) \to \lim_V \mathcal{F}(V)$$

is an equivalence in $\mathcal{C}$. We denote by $\mathsf{Shv}(X;\mathcal{C})$ the ∞-category of $\mathcal{C}$-valued sheaves. If $\mathcal{C}$ is a stable ∞-category, so is $\mathsf{Shv}(X;\mathcal{C})$.

Remark 2.2. Let us spell out the meaning of limits, or more commonly called *homotopy limits*, in the ∞-category Spc of spaces. Let I be a diagram, and $F\colon I \to \mathsf{Spc}$ be a functor. A *cone* over F is a pair (X, η), where X is a space, and η is a morphism from c_X—the functor from I to Spc assuming the constant value X—to F, in the ∞-category of functors Spc^I.

For example, if $I = \bullet \to \bullet \leftarrow \bullet = \Delta^1 \times_{d^0, \Delta^0, d^0} \Delta^1$, then a functor $F\colon I \to \mathsf{Spc}$ is the same thing as choosing three spaces joined by two maps

$$A \to S \leftarrow B. \tag{2.3}$$

A cone (X, η) over F is the same as a diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_A} & A \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{b} & S \end{array},$$

together with a homotopy $h\colon a\eta_A \simeq b\eta_B$. In other words, a cone over F is the same as a functor $F\colon \Delta^1 \times \Delta^1 \to \mathsf{Spc}$ such that the vertices at $(1,0)$, $(0,1)$, and $(1,1)$ are A, B, and S, respectively.

We shall not recall the definition of a limit cone, content with the following characterization. A cone (X, η) is a *limit cone* of F, if and only if for any space T, the natural map between plain sets

$$[T, X]_{\mathsf{Spc}} \to [c_T, F]_{\mathsf{Spc}^I} \tag{2.4}$$

is bijective. Here, for an ∞-category $\mathcal{C}$, $[X, Y]_{\mathcal{C}} = \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y))$ is the set of morphisms in the homotopy category $h\mathcal{C}$. For instance, one particular limit of the pullback diagram (2.3), in the ∞-categorical sense, may be given by a classical formula [3]

$$A \times^h_S B = \{(\alpha, \beta, \gamma) \in A \times B \times S^{\Delta^1} : a(\alpha) = \gamma(0), b(\beta) = \gamma(1)\},$$

together with two projections to A and B. A more functorial way to write it is:

$$A \times^h_S B = A \times_{a, S, \mathrm{ev}_0} S^{\Delta^1} \times_{\mathrm{ev}_1, S, b} B = (A \times B) \times_{S \times S} S^{\Delta^1}.$$

Notably, what happened in the formula is that we replace the space S by a "fatter" but homotopy equivalent space S^{Δ^1}. See Kerodon, `Tag 0106`, and references therein for more discussions.

For any small diagram I, and any functor $F\colon I \to \mathsf{Spc}$, its limit $\lim_I F$ exists in Spc. Moreover, it can be shown that the ∞-category of limit cones of F is trivial (i.e., all mapping spaces of this ∞-category are contractible). Therefore, if (X, η) and (X', η') are two limit cones, then X is equivalent to X' in a *canonical* way (the space of all cone-structure preserving equivalences is contractible).

Example 2.5. Let X be a topological space. Let $\mathcal{J}$ be an injective abelian sheaf in the traditional sense. Then $U \mapsto \mathcal{J}(U)$ is also sheaf valued in $\mathcal{D}(\mathbf{Z})$—the derived ∞-category of abelian groups—in the sense of Definition 2.1. The essence is that the injectivity of $\mathcal{J}$ ensures that the traditional limit $\lim_V \mathcal{J}(V)$ is also the homotopy limit.

Using this observation, if $\mathcal{F}$ is a traditional sheaf of abelian groups, and $\mathcal{F} \hookrightarrow \mathcal{J}^\bullet$ is an injective resolution, then the assignment $U \mapsto R\Gamma(U; \mathcal{F}) := \mathcal{J}^\bullet(U)$ is also a $\mathcal{D}(\mathbf{Z})$-valued sheaf in the sense of Definition 2.1. Sometimes this abuse of notation can be justified. For example, if X is locally contractible, then the space of derived sections $R\Gamma(U; \mathbf{Z}_X)$ is discrete, and thus is equal to the usual section space $\mathbf{Z}_X(U)$. This is the core of many "local arguments".

Definition 2.6. Let $\mathcal{C}$ be an ∞-category. A $\mathcal{C}$-valued *cosheaf* on X is a *covariant* functor $\mathcal{F}$ from the category of open subsets of X into Ab, such that for any open subset U of X, and any open covering $(U_\alpha)_{\alpha \in A}$ of U, the colimit $\operatorname{colim}_V \mathcal{F}(V)$ exists, where V ranges over all open subsets of U which are contained in some U_α, and the canonical map

$$\operatorname*{colim}_V \mathcal{F}(V) \to \mathcal{F}(U)$$

is an equivalence. We denote the the stable ∞-category of $\mathcal{C}$-valued cosheaves as $\mathsf{cShv}(X; \mathcal{C})$. By definition, the category $\mathsf{cShv}(X; \mathcal{C})$ is simply the ∞-category $\mathsf{Shv}(X; \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Example 2.7. A typical example of a cosheaf is the *singular complex*. Consider any topological space X. Then, the mapping $U \mapsto \mathrm{Sing}_*(U)$, where $\mathrm{Sing}_*(U)$ is the simplicial set formed by the singular chains of U, is a cosheaf (valued in the ∞-category Spc) in the sense of Definition 2.6. Proving this fact essentially involves the conventional argument of the excision theorem found in standard textbooks on algebraic topology. Additional information can be found in Kerodon, Tag 012B.

Note that each distinct assignment $U \mapsto \mathrm{Sing}_m(U)$ is *not* a cosheaf in the traditional sense. The inherent flexibility provided by the ∞-categorical language is one of its advantage over the conventional language.

Definition 2.8. Let $\mathcal{F}$ be a $\mathcal{D}(\mathbf{Z})$-valued sheaf, where $\mathcal{D}(\mathbf{Z})$ is the derived ∞-category of abelian groups. Recall that the sheaf cohomology $H^*(X;\mathcal{F})$ is defined to be the usual cohomology of $\mathcal{F}(X)$. For a $\mathcal{D}(\mathbf{Z})$-valued cosheaf, we can equally define the *cosheaf homology* via the formula

$$H_*(X;\mathcal{F}) := H^{-*}\mathcal{F}(X).$$

Example 2.9. Let X be a space. Let $C_*(U)$ be the chain complex associated to the simplicial abelian group $\mathbf{Z}\mathrm{Sing}_*(U)$. By Example 2.7, the assignment $U \mapsto C_*(U)$ is a cosheaf on X valued in $\mathcal{D}(\mathbf{Z})$. Its cosheaf homology is simply the singular homology of space X.

Definition 2.10. Let $f\colon X \to Y$ be a map of spaces. Then we may define the *direct image* of a $\mathcal{C}$-valued sheaf $\mathcal{F}$ by the usual formula

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

Under some mild conditions, f_* has a left adjoint f^*. For example, this is the case when $\mathcal{C}$ is the ∞-category of spaces or spectra, or the stable ∞-category of abelian groups or R-modules for some ring R.

Similarly, we may define the direct image of a $\mathcal{C}$-valued cosheaf $\mathcal{F}$ by the same formula

$$f_\dagger\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

Since $\mathsf{cShv}(X;\mathcal{C})$ is the opposite category of $\mathsf{Shv}(X;\mathcal{C}^{\mathrm{op}})$, in many preferable situations $f_\dagger$ admits a right adjoint $f^\dagger$.

3. Covariant Verdier duality: statement

We now present the covariant Verdier duality theorem and discuss some of its consequences.

Definition 3.1. Let X be a locally compact Hausdorff space. Let $\mathcal{C}$ be an ∞-category with small limits and colimits. Let $\mathcal{F}$ be a $\mathcal{C}$-valued sheaf.

For a compact subspace K of X, we set

$$\mathcal{F}(K) = \operatorname*{colim}_{K\subset V}\mathcal{F}(V),$$

the colimit being taken over all open subsets V of X containing K.

We define the *space of sections of $\mathcal{F}$ over* U *with compact supports* as

$$\Gamma_c(U;\mathcal{F}) = \mathcal{F}_c(U) = \operatorname*{colim}_{K\subset U}\Gamma_K(U;\mathcal{F}),$$

where $\Gamma_K(U;\mathcal{F})$ is the homotopy kernel of $\mathcal{F}(U) \to \mathcal{F}(K)$. We shall show later that $\mathcal{F}_c$ is indeed a cosheaf on X (cf. 4.2, 4.3).

Recollection 3.2 (Stable ∞-categories). The theory of stable ∞-categories is recorded in the on-going book project *Higher algebra* of Lurie. The first chapter is a very good introduction to the subject. Here is an oversimplified summary. An ∞-category $\mathcal{C}$ is said to be a *stable* ∞-category, if it has all finite limits and colimits, and any square in $\mathcal{C}$ is cartesian if and only if it is cocartesian. In particular, a stable ∞-category $\mathcal{C}$ has an object 0 which is both initial and terminal. An *exact triangle* in $\mathcal{C}$ is a cartesian (hence also cocartesian) square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

If $\mathcal{C}$ and $\mathcal{C}'$ are stable ∞-categories, and $F\colon \mathcal{C} \to \mathcal{C}'$ is a functor, then we say F is *exact*, if it commutes with finite limits (this is equivalent to requiring it to commute with finite colimits).

If $\mathcal{C}$ is a stable ∞-category, then it can be shown that $h\mathcal{C}$ is a triangulated category:

- ◇ The shift functor $X \mapsto X[1]$ is given by the suspension ΣX, defined by

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

- ◇ Distinguished triangles $X \to Y \to Z \to X[1]$ in $h\mathcal{C}$ comes from exact triangles defined as above.

Typical examples of stable ∞-categories are Spt, the category of spectra, $\mathcal{D}(R)$, the derived ∞-category of R-modules, and the bounded variants $\mathcal{D}^+(R)$, $\mathcal{D}^-(R)$, $\mathcal{D}^b(R)$ of $\mathcal{D}(R)$. If I is a simplicial set, and $\mathcal{C}$ is a stable ∞-category, then the functor category $\mathcal{C}^I$ is a stable ∞-category. If X is a space, and $\mathcal{C}$ is a stable ∞-category, then $\mathsf{Shv}(X; \mathcal{C})$ is a stable ∞-category.

THEOREM 3.3. *Let* X *be a locally compact Hausdorff space. Let* $\mathcal{C}$ *be a* stable ∞*-category with small limits and colimits. Then the functor* [4]

$$\mathbf{V}_{X;\mathcal{C}}\colon \mathsf{Shv}(X; \mathcal{C}) \to \mathsf{cShv}(X; \mathcal{C}), \quad \mathcal{F} \mapsto \mathcal{F}_c$$

called covariant Verdier duality functor*, is an exact equivalence of stable* ∞*-categories.*

The proof of Theorem 3.3 will be given at the end of this section. For now, let us discuss some implications of this theorem.

DEFINITION 3.4 (Extraordinary direct and inverse images). Let $f\colon X \to Y$ be a continuous map between locally compact Hausdorff spaces. We define

$$f_! = \mathbf{V}_{Y;\mathcal{C}}^{-1} \circ f_\dagger \circ \mathbf{V}_{X;\mathcal{C}}\colon \mathsf{Shv}(X; \mathcal{C}) \to \mathsf{Shv}(Y; \mathcal{C}),$$

and call it the *extraordinary direct image functor*.

Suppose that $f_\dagger$ has a right adjoint $f^\dagger$, we define

$$f^! = \mathbf{V}_{X;\mathcal{C}}^{-1} \circ f^\dagger \circ \mathbf{V}_{Y;\mathcal{C}}\colon \mathsf{Shv}(Y; \mathcal{C}) \to \mathsf{Shv}(X; \mathcal{C}),$$

and call it the *extraordinary inverse image functor*.

Since $f^\dagger$ is a right adjoint to $f_\dagger$, and since $\mathbf{V}_{X;\mathcal{C}}$ and $\mathbf{V}_{Y;\mathcal{C}}$ are equivalences of categories, we find that $f^!$ is a right adjoint to $f_!$: for $\mathcal{F} \in \mathsf{Shv}(X; \mathcal{C})$ and $\mathcal{G} \in \mathsf{Shv}(Y; \mathcal{C})$, we have

$$\mathrm{Map}_{\mathsf{Shv}(Y;\mathcal{C})}(f_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Map}_{\mathsf{Shv}(X;\mathcal{C})}(\mathcal{F}, f^!\mathcal{G}), \tag{3.5}$$

bifunctorial with respect to $\mathcal{F}$ and $\mathcal{G}$.

REMARK 3.6 (Extraordinary direct image and direct image with proper support). When $\mathcal{C}$ is the derived ∞-category of abelian groups, we can describe $f_!\mathcal{F}$ in classical terms: $f_!\mathcal{F}(U)$ consists of sections of $\mathcal{F}$ over $f^{-1}(U)$ whose supports are proper over U. To see that this classical description coincides with the one defined earlier, we only need to verify that they have the same compactly supported sections over U, which is evident. From these discussions, we see (3.5) implies an "unbounded" version of the relative Verdier duality (1.7) by passing to the homotopy category. We remark that if X is paracompact of finite covering dimension (this is the situation where the classical Verdier duality applies), or a CW-complex, then $\mathsf{hShv}(X; \mathcal{D}(\mathbf{Z}))$ is equivalent, as a triangulated category, to $\mathsf{D}(X; \mathbf{Z})$, the unbounded derived category of abelian sheaves on X; but in general the two need not be equivalent. See Note 2 for more discussions on this topic.

It is also worth noting that our discussions above do not require any assumptions regarding the covering dimension of the spaces involved. Without the finite dimensionality assumption one should not anticipate $f^!(\mathsf{D}^+(Y)) \subset \mathsf{D}^+(X)$. Indeed, if we consider the situation where Y is a point, X is infinite dimensional, and $f = a_X$ is the canonical map, then the *dualizing complex* $\omega_X = a_X^!\mathbf{Z}$, is not bounded from below, since it can be represented by the complex of Borel–Moore chains (to be elaborated in Remark 3.11 below).

Another drawback for the lack of finite-dimensionality is that the homotopy category of $\mathsf{Shv}(X;\mathcal{C})$ is not equivalent to the classical unbounded derived category.

Remark 3.7 (Compactly supported cohomology and cosheaf homology). Assume that Y is a singleton. In this situation, $\mathsf{Shv}(Y;\mathcal{C}) = \mathcal{C} = \mathsf{cShv}(Y;\mathcal{C})$, and $\mathbf{V}_{Y;\mathcal{C}}$ becomes the identity functor. As a consequence, for any $\mathcal{C}$-valued sheaf $\mathcal{F}$ on X, we have $f_!\mathcal{F} = \mathcal{F}_c(X) = \Gamma_c(X;\mathcal{F})$. In particular, if $\mathcal{C}$ is the derived ∞-category of abelian groups, the cohomology of $f_!\mathcal{F}$ coincides with the *compactly supported cohomology* of the sheaf $\mathcal{F}$. Taking cohomology, we obtain the following relationship:

$$H_c^{-*}(X;\mathcal{F}) \simeq H_*(X;\mathcal{F}_c).$$

Here, the right-hand side represents the cosheaf homology functor, as discussed in Definition 2.8.

Remark 3.8 (Constant cosheaf). Let X be a locally compact topological space. Let $\mathbf{Z}_X$ be the constant cosheaf on X valued in $\mathbf{Z}$: $\mathbf{Z}_X = a_X^\dagger\mathbf{Z}$, where $a_X\colon X \to \mathrm{pt}$ is the canonical map to a singleton.[1] Since a $\mathcal{C}$-valued cosheaf is none other than a $\mathcal{C}^{\mathrm{op}}$-valued sheaf, the usual formula gives $\mathbf{Z}_X(U) = \operatorname{colim}_{V\subset U}\mathbf{Z}$, where V runs through all open subsets contained in U. If X a locally contractible paracompact space, and $U \mapsto C_*(U)$ is the cosheaf of singular chains considered in Example 2.9, then the natural map $\mathbf{Z}_X \to C_*$ is an equivalence, since it is an equivalence on contractible open sets.

With these remarks we can now define the *dualizing complex* of a locally compact space.

Definition 3.9 (Dualizing complex). Let X be a locally compact space. We define ω_X to be the $\mathcal{D}(\mathbf{Z})$-valued sheaf satisfying $(\omega_X)_c = \mathbf{Z}_X$. The object ω_X is unique up to equivalences, and we refer to it as the *dualizing complex* of X.

In view of Definition 3.4 and Remark 3.8, we could have defined ω_X using the formula $\omega_X = a_X^!\mathbf{Z}$.

[5] Definition 3.10 (Borel–Moore homology). Let X be a locally compact space. We define the i^{th} *Borel–Moore homology* of X by

$$H_i^{\mathrm{BM}}(X) := H^{-i}(X;\omega_X).$$

Remark 3.11 (Relation with Borel–Moore chains). We introduced the complex of geometric Borel–Moore chains in Remark 1.5, and asserted there that $\omega_X(U)$ is the complex of geometric chains whose supports are contained in U. In this remark we explain why this is consistent with the current framework.

Let X be a PL-space of dimension n. The mapping which sends U to the complex of geometric k-chains whose supports are contained in U is a *flasque* (en: *flabby*) sheaf, denoted by $\mathcal{C}_k^{\mathrm{BM}}(U)$. Thus we get an object $\mathcal{C}_\bullet^{\mathrm{BM}}$ in $D^b(X)$ concentrated in degrees $[-n, 0]$.

Since a Borel–Moore simplicial chain has compact support if and only if is a *finite* linear combination of simplices, the complex of compactly supported sections of $\mathcal{C}_\bullet^{\mathrm{BM}}(U)$ admits a chain map to $C_\bullet(U)$, the complex of singular chains. By standard topology, this inclusion map is a homotopy equivalence. Thus, $\mathcal{C}_c^{\mathrm{BM}}$ is isomorphic to the cosheaf $U \mapsto C_*(U)$ described in Example 2.7. Since $U \mapsto C_*(U)$ is isomorphic to the constant cosheaf $\mathbf{Z}_X$ in $\mathsf{cShc}(X;\mathcal{D}(\mathbf{Z}))$ (Remark 3.8), and since ω_X is defined as the sheaf satisfying $\omega_{X,c} = \mathbf{Z}_X$, we conclude that $\omega_X = \mathcal{C}^{\mathrm{BM}}$.

Remark 3.12 (Classical Poincaré duality). One can show that

$$H_i^{\mathrm{BM}}(\mathbf{R}^n) = \begin{cases} 0 & i \neq n \\ \mathbf{Z} & i = n. \end{cases}$$

This shows that if X is a manifold of pure dimension n, then ω_X is of the form $or_X[n]$ for some sheaf or_X, and or_X is locally isomorphic to the constant sheaf $\mathbf{Z}$. We call or_X the *orientation sheaf* of the manifold. We say X is *orientable* if or_X is globally isomorphic to $\mathbf{Z}_X$. By Remark 3.7, we see, for any manifold X, we have an isomorphism

$$H_c^{n-i}(X;or_X) \simeq H_i(X;\mathbf{Z}),$$

this recovers the classical Poincaré duality (1.1).

[1] We apologize for using $\mathbf{Z}_X$ to denote both the constant sheaf and constant cosheaf. They are totally different beasts, but we run out of reasonable symbols. Fortunately, I don't think the cosheaf $\mathbf{Z}_X$ will appear again.

Remark 3.13 (The *contravariant* Verdier duality). Let $\mathcal{C}$ be a stable ∞-category with small limits and colimits. Suppose $\mathcal{C}$ is equipped with a contravariant endofunctor

$$M \mapsto M^\vee : \mathcal{C}^{op} \to \mathcal{C}.$$

For instance, we can consider $\mathcal{C} = \mathcal{D}(R)$, the derived ∞-category of R-modules, where R is a commutative ring, and $M^\vee = RHom_R(M, R)$.

Let $\mathcal{F}$ be a $\mathcal{C}$-valued cosheaf on X. We can define a $\mathcal{C}$-valued sheaf $\mathcal{F}^\vee$ on X by setting $\mathcal{F}^\vee(U) = \mathcal{F}(U)^\vee$. The composition $\mathcal{F} \mapsto \mathbf{V}_{X;\mathcal{C}}(\mathcal{F})^\vee : \mathsf{Shv}(X;\mathcal{C})^{op} \to \mathsf{Shv}(X;\mathcal{C})$ then becomes a *contravariant functor*. We denote $\mathbf{V}_{X;\mathcal{C}}(\mathcal{F})^\vee$ by $\mathbf{D}_X(\mathcal{F})$ or simply $\mathbf{D}(\mathcal{F})$, and refer to this sheaf as the *Verdier dual* of $\mathcal{F}$,

Let us now specialize to $\mathcal{C} = \mathcal{D}(R)$, the derived ∞-category of R-modules, where $\vee$ is the usual dual functor of R-modules: $M^\vee = RHom_R(M, R)$. Since $\mathcal{F}^\vee(U) = \mathcal{F}(U)^\vee$, we have

$$RHom(H_*(X;\mathcal{F}), R) = H^*(X;\mathcal{F}^\vee).$$

It follows that for any $\mathcal{D}(R)$-valued sheaf $\mathcal{G}$ on X, we have

$$\boxed{RHom(H_c^{-*}(X;\mathcal{G}); R) = H^*(X;\mathbf{D}(\mathcal{G}))}.$$

This recovers the traditional Verdier duality.

4. Covariant Verdier duality: proof

In this paragraph, let $\mathcal{C}$ be a stable ∞-category with small limits and colimits. Then we know that both in $\mathcal{C}$ and its opposite category $\mathcal{C}^{op}$, filtered colimits are exact. We also fix a locally compact Hausdorff topological space X.

Definition 4.1 ($\mathcal{K}$-sheaves). Let $\mathcal{K}(X)$ be the partially ordered set consisting of compact subspaces of X. A functor $\mathcal{F}: \mathcal{K}(X)^{op} \to \mathcal{C}$ is called a *$\mathcal{K}$-sheaf* on X, if it satisfies the following properties

(i) $\mathcal{F}(\emptyset) = 0$,

(ii) for any two compact subspaces K and K′ of X, the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(K \cup K') & \longrightarrow & \mathcal{F}(K) \\ \downarrow & & \downarrow \\ \mathcal{F}(K') & \longrightarrow & \mathcal{F}(K \cap K') \end{array}$$

is cartesian,

(iii) for every K, the natural map

$$\operatorname*{colim}_{K \Subset K'} \mathcal{F}(K') \to \mathcal{F}(K)$$

is an equivalence; here $K \Subset K'$ means K is contained in the interior of K′, and the colimit is taken over all K′ in $\mathcal{K}(X)$ such that $K \Subset K'$.

The category of $\mathcal{K}$-sheaves on X is denoted by $\mathsf{Shv}_{\mathcal{K}}(X;\mathcal{C})$.

Construction 4.2 (Equivalence between $\mathsf{Shv}_{\mathcal{K}}$ and Shv). If $\mathcal{F}$ is a $\mathcal{K}$-sheaf on X, then we define a sheaf $\psi(\mathcal{F})$ on X by

$$\psi(\mathcal{F})(U) = \lim_{K \subset U} \mathcal{F}(K).$$

If $\mathcal{G}$ is a sheaf on X, then we define a $\mathcal{K}$-sheaf $\theta(\mathcal{G})$ on X by

$$\theta(\mathcal{G})(K) = \operatorname*{colim}_{K \subset U} \mathcal{G}(U).$$

Since X is assumed to be locally compact and Hausdorff, one can show that the functor $\psi: \mathsf{Shv}_{\mathcal{K}}(X;\mathcal{C}) \to \mathsf{Shv}(X;\mathcal{C})$ is an equivalence of ∞-categories, and θ is a quasi-inverse to ψ.

Similarly, if $\mathcal{F}$ is a $\mathcal{K}$-cosheaf on X, we define

$$\psi(\mathcal{F})(U) = \operatorname*{colim}_{K \subset U} \mathcal{F}(K),$$

and if $\mathcal{G}$ is a cosheaf on X, we define

$$\theta(\mathcal{G})(K) = \lim_{K \subset U} \mathcal{G}(U).$$

The functor ψ establishes an equivalence between $\mathsf{cShv}_{\mathcal{K}}(X;\mathcal{C})$ and $\mathsf{cShv}(X;\mathcal{C})$, with quasi-inverse θ.

PROPOSITION 4.3. *Let $\mathcal{F}$ be an object of $\mathsf{Shv}(X;\mathcal{C})$. Let K be a compact subset of X. Define $\Gamma_K(X;\mathcal{F})$ as the homotopy fiber of $\Gamma(X;\mathcal{F}) \to \Gamma(X-K;\mathcal{F})$. Then the assignment $K \mapsto \Gamma_K(X;\mathcal{F})$ is a $\mathcal{K}$-cosheaf on X.*

PROOF. The dual condition of Definition 4.1(i) is clear. Let K and K' be two compact subspaces of K. Then the square

$$\begin{array}{ccc} \Gamma_{K\cap K'}(X;\mathcal{F}) & \longrightarrow & \Gamma_K(K;\mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma_{K'}(X;\mathcal{F}) & \longrightarrow & \Gamma_{K\cup K'}(K;\mathcal{F}) \end{array} \tag{4.4}$$

is the homotopy fiber of the obvious map from

$$\begin{array}{ccc} \mathcal{F}(X) & = & \mathcal{F}(X) \\ \| & & \| \\ \mathcal{F}(X) & = & \mathcal{F}(X) \end{array} \quad \text{to} \quad \begin{array}{ccc} \mathcal{F}(X-(K\cap K')) & \longrightarrow & \mathcal{F}(X-K) \\ \downarrow & & \downarrow \\ \mathcal{F}(X-K') & \longrightarrow & \mathcal{F}(X-(K\cup K')). \end{array}$$

The first square is trivially cartesian, the second square is cartesian since $X-(K\cap K')$ equals $(X-K)\cup(X-K')$. Therefore, (4.4) is cartesian as well. This verifies the dual of Item 4.1(ii). Finally, we need to check the dual of Item 4.1(iii), that is, for any compact subspace K of X, the natural map

$$\Gamma_K(X;\mathcal{F}) \to \lim_{K\Subset K'} \Gamma_{K'}(X;\mathcal{F})$$

is an equivalence. For this we simply invoke the definition:

$$\begin{array}{ccc} \Gamma_K(X;\mathcal{F}) & \longrightarrow & \lim_{K\Subset K'} \Gamma_{K'}(X;\mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{F}(X) & = & \lim_{K\Subset K'} \mathcal{F}(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(X-K) & \longrightarrow & \lim_{K\Subset K'} \mathcal{F}(X-K'). \end{array}$$

In the diagram, the two vertical sequence are exact. The left one is exact by definition; the right one is exact is because $\mathcal{C}$ is stable, in which cofiltered limits are exact. The middle arrow is an equivalence since the limit is taken over a cofiltered system, which is contractible; the lower horizontal arrow is an equivalence since $X-K'$ form an open covering of $X-K$, and the equivalence follows from the sheaf axiom. Therefore, the upper horizontal arrow is necessarily an equivalence. □

THEOREM 4.5. *The functor $\mathbf{V}_{X;\mathcal{C}}\colon \mathsf{Shv}(X;\mathcal{C}) \to \mathsf{cShv}(X;\mathcal{C})$ induced by the functor defined in Proposition 4.3 is an equivalence of stable ∞-categories.*

PROOF. Let us make the functor explicit. Start with a $\mathcal{C}$-valued sheaf $\mathcal{F}$ on X. It defines a $\mathcal{K}$-cosheaf by Proposition 4.3, and thus gives rise to a $\mathcal{C}$-valued cosheaf by the functor ψ defined in 4.2. Thus, for an open set U, $\mathbf{V}_{X;\mathcal{C}}(\mathcal{F})(U) = \operatorname{colim}_{K\subset U}\Gamma_K(X;\mathcal{F}) = \Gamma_c(U;\mathcal{F})$. To check that $\mathbf{V}_{X;\mathcal{C}}$ is an equivalence, we only need to check that $\mathbf{V}_{X;\mathcal{C}^{\mathrm{op}}}(\mathbf{V}_{X;\mathcal{C}}(\mathcal{F}))$ is equivalent to $\mathcal{F}$ on compact subspaces, since the other composition follows from symmetry. Spelling out everything, we are reduced to proving that for any compact subspace K of X, the following natural sequence is exact:

$$\Gamma_c(X-K;\mathcal{F}) \to \Gamma_c(X;\mathcal{F}) \to \operatorname*{colim}_{K\subset U}\mathcal{F}(U) = \mathcal{F}(K). \tag{4.6}$$

In ordinary sheaf theory, this is well-known. But since we have to work with some added generality, we nevertheless supply a proof of it. In order to prove the exactness of (4.6), we shall first prove the following

Lemma 4.7. *Let* U *be an open subset of* X*. Let* L *be a compact subspace of* X *containing* U*. Then we have a exact triangle*

$$\begin{array}{ccc} \Gamma_{L-U}(X;\mathcal{F}) & \longrightarrow & \Gamma_L(X;\mathcal{F}) \\ \downarrow & \square & \downarrow \\ 0 & \longrightarrow & \Gamma(U;\mathcal{F}). \end{array}$$

Proof of Lemma 4.7. We notice that $X-(L-U)$ is a disjoint union of two open subsets: $U \amalg (X-L)$. Therefore, $\mathcal{F}(X-(L-U)) = \mathcal{F}(U) \oplus \mathcal{F}(X-L)$. It follows that the homotopy kernel of

$$\Gamma(X-(L-U);\mathcal{F}) \to \Gamma(X-L;\mathcal{F})$$

is identified with $\mathcal{F}(U)$. With this in mind we can consider the following commutative diagram

$$\begin{array}{ccccc} \Gamma_{L-U}(X;\mathcal{F}) & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \\ \Gamma_L(X;\mathcal{F}) & \longrightarrow & \Gamma(U;\mathcal{F}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(X;\mathcal{F}) & \longrightarrow & \Gamma(X-(L-U);\mathcal{F}) & \longrightarrow & \Gamma(X-L;\mathcal{F}). \end{array}$$

In the diagram, the right lower rectangle is cartesian by our discussion above. The lower large rectangle is cartesian by definition. It follows that the lower left rectangle is cartesian. Since the upper left rectangle is cartesian by definition, we conclude that the right large rectangle is cartesian. This gives the desired the result. □

Going back to the proof of the exactness of (4.6), we notice that $\Gamma_c(X-K)$ equals

$$\operatorname*{colim}_{U\subset L}\operatorname*{colim}_{K\subset U}\Gamma_{L-U}(X;\mathcal{F})$$

and $\Gamma_c(X;\mathcal{F})$ equals

$$\operatorname*{colim}_{U\subset L}\operatorname*{colim}_{K\subset U}\Gamma_{L}(X;\mathcal{F}).$$

This completes the proof. □

Notes

1. A paracompact topological space X is said to have *covering dimension* $\leq n$, if the following condition is satisfied: any open covering $\{U_\alpha\}_{\alpha\in I}$ of X, has a refinement $\{V_\beta\}_{\beta\in J}$, such that $\bigcap_{\beta\in K} V_\beta = \emptyset$, where K is any subset of J whose cardinality equals $n+1$. For example, any complex quasi-projective variety which can be embedded into $\mathbf{P}^n_{\mathbf{C}}$ (equipped with the classical topology) has covering dimension $\leq n$.

2. Let Spc be the ∞-category of spaces. In the following, denote the ∞-category of Spc-valued sheaves by Shv(X), and the ∞-category of $\mathcal{D}(\mathbf{Z})$-valued sheaves by $\mathsf{Shv}(X;\mathbf{Z})$. Then Shv(X) is an ∞-topoi.

On a general space, there could exist $\mathcal{F} \in \mathsf{Shv}(X)$, such that all the stalks $\mathcal{F}_x$ are weakly contractible, but $\mathcal{F}$ is not isomorphic to the trivial sheaf. Similar examples can also be constructed in $\mathsf{Shv}(X;\mathbf{Z})$. One such space is the Hilbert cube $X = \prod_{\mathbf{N}}[0,1]$ (See Lurie 2009, Counterexample 6.5.4.2; see also Note 5). This renders $h\mathsf{Shv}(X;\mathbf{Z}) \not\approx D(X;\mathbf{Z})$, where $D(X;\mathbf{Z})$ is the unbounded derived category of complexes of abelian sheaves (see Spaltenstein 1988).

The ∞-categorical enhancement of the unbounded derived category as defined in ibid. is the ∞-category of *hypersheaves*, detailed below.

Let W be the set of morphisms of Shv(X) are stalk-wise weak equivalences. Then the ∞-category of local objects with respect to W are called *hypersheaves*, denoted as $\widehat{\mathsf{Shv}}(X)$. It can be shown that $\widehat{\mathsf{Shv}}(X)$ is equivalent to the localization $\mathsf{Shv}(X)[W^{-1}]$ (the "hypercompletion" of Shv(X)), and we get a pair of adjoint functors

$$L \dashv \iota: \quad \widehat{\mathsf{Shv}}(X) \underset{L}{\overset{\iota}{\rightleftarrows}} \mathsf{Shv}(X).$$

Also, for a sheaf $\mathcal{F}$ to be a hypersheaf, it is necessary and sufficient that $\mathcal{F}$ satisfies a more stringent descent condition: for any open set U and any *hypercovering* $\mathfrak{U} \to U$, $\mathcal{F}(U) \xrightarrow{\sim} \lim \mathcal{F}(\mathfrak{U})$.

An abelian analogue $\widehat{\mathsf{Shv}}(X;\mathbf{Z})$, the "hypercompletion" of $\mathsf{Shv}(X;\mathbf{Z})$, can also be constructed. It can be shown that

$$\mathsf{h}\widehat{\mathsf{Shv}}(X;\mathbf{Z}) \simeq \mathsf{D}(X;\mathbf{Z}).$$

Hypersheaves are discussed in more detail in Lurie 2009, Section 6.5.3. For a comprehensive comparison between sheaves and hypersheaves, particularly when it comes to determining which concept is more suitable for specific scenarios, the reader is directed to ibid., Section 6.5.4. An influential classical reference on this topic is Dugger, Hollander, and Isaksen 2004. We only remark that while $\widehat{\mathsf{Shv}}(X)$ aligns more closely with the classical unbounded derived category, there are cases where $\mathsf{Shv}(X)$ is preferable for certain applications. For instance, the proper base change theorem (see *Proper base change in topology*, Note 2) holds for sheaves, but may not hold for hypersheaves in general.

There are spaces on which we do not need to worry about the distinctions between sheaves and hypersheaves. We say X is *hypercomplete* (or rather, the ∞-topos $\mathsf{Shv}(X)$ is hypercomplete), if the inclusion functor $\iota\colon \widehat{\mathsf{Shv}}(X) \to \mathsf{Shv}(X)$ is an equivalence of ∞-categories. The Hilbert cube is thus a compact Hausdorff space that is not hypercomplete. Here are some criteria for X to be hypercomplete:

- ◇ Any paracompact (Hausdorff) topological space of finite covering dimension is hypercomplete (Lurie 2009, Corollary 7.2.1.12, Theorem 7.2.3.6). In fact, Lurie introduced a new notion called *homotopy dimension*: an ∞-topos is said to have homotopy dimension $\leq n$ if any n-connective object of the topos has a section. Lurie showed that any ∞-topos with finite homotopy dimension is hypercomplete (ibid., Corollary 7.2.1.12, attributed to Jardine).
- ◇ Let X be a paracompact space, $X_0 \subset X_1 \subset \cdots$ a sequence of closed subspaces. If $X = \bigcup X_i$, and if a subset U of X is open if and only if $U \cap X_i$ is open in X_i for all i, then
  $$[X_i \text{ are hypercomplete for all } i] \Rightarrow [X \text{ is hypercomplete}].$$
  This follows from
  - – ibid., Proposition 7.1.5.8, which asserts that $\mathsf{Shv}(X) = \operatorname{colim} \mathsf{Shv}(X_i)$ in the ∞-category of ∞-topoi;
  - – the fact that hypercomplete ∞-topoi form a coreflexive ∞-subcategory of the ∞-category of ∞-topoi (ibid., Proposition 6.5.2.13); and
  - – the fact that the inclusion functor of a coreflexive subcategory creates colimits (this is the dual statement of Kerodon, `Tag 03XY`), hence the colimit of hypercomplete ∞-topoi is hypercomplete.

  In particular, *any CW complex is hypercomplete.*[2]
- ◇ If $\mathsf{Shv}(X)$ has enough points, then X is hypercomplete (ibid., Combine Remark 6.5.2.2 and Proposition 6.5.2.14).

3. Let $F : [A \to S \leftarrow B]$ be a pullback diagram in the ∞-category of spaces. Then we have discussed the notion of a homotopy pullback in Remark 2.2. There, homotopy pullback is characterized as a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{\ a\ } & S \end{array}$$

satisfying the property (2.4), namely, for any space T, we have $[T,X]_{\mathsf{Spc}}$ is *isomorphic* to $[c_T, F]_{\mathsf{Spc}^I}$ in the functor category. As a warning, we remark that $[c_T, F]_{\mathsf{Spc}^I}$ is *not* the fiber product $[T,A]_{\mathsf{Spc}} \times_{[T,S]_{\mathsf{Spc}}} [T,B]_{\mathsf{Spc}}$. The latter set is a quotient set $M/\sim$, where M is the set of maps $\xi_A\colon T \to A$ and $\xi_B\colon T \to B$, such that $a \circ \xi_A$ is homotopic to $b \circ \xi_B$ (but we don't remember the information of homotopy); the equivalence relation is given by the homotopy relation on each component. The set $[c_T, F]_{\mathsf{Spc}^I}$ is also quotient set $N/\sim$, where N is the set of triples (ξ_A, ξ_B, h), where ξ_A and ξ_B are as before, and h is a specific homotopy from $a \circ \xi_A$ to $b \circ \xi_B$. The pair (ξ_A, ξ_B, h) and (ξ'_A, ξ'_B, h') are equivalent if and only if there exists homotopies $\xi_A \simeq \xi'_A$, $\xi_B \simeq \xi'_B$, such that, they form a homotopy coherent diagram together with the given homotopies

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[2] Recall: a CW complex is locally compact if and only if it is locally finite; but it is always paracompact. If we want to apply Verdier duality to CW complexes, we have to manifestly work with CW complexes whose connected components are finite dimensional, a bit boring!

h and h'. This is a *stronger condition* than simply saying $\xi_A \simeq \xi'_A$, $\xi_B \simeq \xi'_B$, and they commute with a, b up to homotopy. It follows that $[T,A]_{\mathsf{Spc}} \times_{[T,S]_{\mathsf{Spc}}} [T,B]_{\mathsf{Spc}}$ can be obtained from N by quotienting out a coarser equivalence relation, and we have a *surjective* map

$$[c_T, F]_{\mathsf{Spc}^I} \twoheadrightarrow [T,A]_{\mathsf{Spc}} \times_{[T,S]_{\mathsf{Spc}}} [T,B]_{\mathsf{Spc}}.$$

4. The history of the covariant Verdier duality theorem is not entirely clear to me. Using our own convention, the duality functor $\mathcal{F} \mapsto \mathbf{D}\mathcal{F}$ appeared in Borel and Moore 1960 is explicitly defined by $\mathcal{F} \mapsto \mathcal{F}_c^\vee$ (see Remark 3.13). Beware that in ibid., constructions of sheaves are always performed after taking a Godement resolution, thus are automatically "derived". However, Borel and Moore did not seem to realize that $\mathcal{F} \mapsto \mathcal{F}_c$ is an equivalence. According to Justin Curry's very influential 2014 thesis, Schneider 1998, "Verdier duality on the building", seems to be the first written document that explicitly proved a formulation of this theorem; and Robert MacPherson conjectured a "celluar" version of the covariant Verdier duality should hold in a lecture talk dated 2012. Curry 2014, Theorem 12.2.1 confirmed MacPherson's conjecture.

The proof of the covariant Verdier duality mostly follows the presentations given in M. Volpe's dissertation, and J. Lurie's book Higher Algebra.

5. Borel–Moore homology was initially introduced by Borel and Moore 1960 through the framework of sheaves. In ibid., the duality functor $\mathcal{G} \mapsto \mathbf{D}\mathcal{G}$ (see Remark 3.13) is introduced for any sheaf of R-modules and the identification

$$\mathrm{RHom}_R(R\Gamma_c(X;\mathcal{G}), R) \simeq R\Gamma(X;\mathbf{D}\mathcal{G})$$

is proved. In fact, Borel–Moore homology (with constant coefficient R) is defined as the cohomology of $\mathbf{D}R_X$.

An alternative formulation later emerged, using locally finite formal linear combinations of singular chains. In the context of piecewise-linear (PL) spaces, the computation of Borel–Moore homology becomes more manageable through the use of simplicial chains: for every admissible triangulation of a PL space, the chain complex of infinite linear combinations of the simplices from its *barycentric subdivision* can yield the correct answer. This proposition can be found in the appendix of MacPherson and Vilonen 1986, authored by Goresky and MacPherson. The necessity of taking a barycentric subdivision is discussed in Remark 2 of that appendix.

When $X = \prod_{\mathbf{N}}[0,1]$ is the Hilbert cube, the sheaf ω_X computing its Borel–Moore homology is an example of a sheaf but not a hypersheaf (See Note 2). Here is the reason (Lurie 2009, Counterexample 6.5.4.8):

- ◇ Every point of X has a neighborhood of homeomorphic to $U = X \times [0,1[$. Therefore one can construct a hypercovering $\mathfrak{U}$ of X using open sets homeomorphic to U.
- ◇ One shows that the Borel–Moore homology of U is zero.
- ◇ If ω_X were to satisfy descent for hypercoverings, then $\omega_X(X) = \lim_{V\in\mathfrak{U}} \omega_X(V)$ would be the limit of the zero group, whence zero.
- ◇ But $H_0^{BM}(X)$ is nonzero: since X is a compact and Hausdorff, we have $H_0^{BM}(X) \approx H_0(X) \neq \{0\}$.

References

Borel, A. and J. C. Moore (1960), *Homology theory for locally compact spaces*, Michigan Math. J. Vol. 7, pp. 137–159 (cit. on pp. 1, 11).

Bott, Raoul and Loring W. Tu (1982), *Differential forms in algebraic topology*, vol. 82, Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, pp. xiv+331 (cit. on p. 1).

Curry, Justin Michael (2014), *Sheaves, cosheaves and applications*, Thesis (Ph.D.)–University of Pennsylvania, ProQuest LLC, Ann Arbor, MI, p. 317 (cit. on p. 11).

Dugger, Daniel, Sharon Hollander, and Daniel C. Isaksen (2004), *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. Vol. 136, no. 1, pp. 9–51 (cit. on p. 10).

Gelfand, Sergei I. and Yuri I. Manin (2003), *Methods of homological algebra*, Second, Springer Monographs in Mathematics, Springer-Verlag, Berlin, pp. xx+372 (cit. on p. 2).

Hatcher, Allen (2002), *Algebraic topology*, Cambridge University Press, Cambridge, pp. xii+544 (cit. on p. 1).

Kashiwara, Masaki and Pierre Schapira (1990), *Sheaves on manifolds*, vol. 292, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], With a chapter in French by Christian Houzel, Springer-Verlag, Berlin, pp. x+512 (cit. on p. 2).

Lurie, Jacob (2009), *Higher topos theory*, vol. 170, Annals of Mathematics Studies, Princeton University Press, pp. xviii+925 (cit. on pp. 9–11).

MacPherson, Robert and Kari Vilonen (1986), *Elementary construction of perverse sheaves*, Invent. Math. Vol. 84, no. 2, pp. 403–435 (cit. on p. 11).

Schneider, Peter (1998), *Verdier duality on the building*, J. Reine Angew. Math. Vol. 494, Dedicated to Martin Kneser on the occasion of his 70th birthday, pp. 205–218 (cit. on p. 11).

Spaltenstein, N. (1988), *Resolutions of unbounded complexes*, Compositio Math. Vol. 65, no. 2, pp. 121–154 (cit. on p. 9).

Verdier, Jean-Louis (1966), *Dualité dans la cohomologie des espaces localement compacts*, in: *Séminaire Bourbaki, Vol. 9*, Soc. Math. France, Paris, Exp. No. 300, 337–349 (cit. on pp. 1, 2).

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