Proper base change in topology

Puncturing inflates the topology.

Contents

The goal of these notes is to state and prove a significant theorem in the sheaf theory, the proper base change theorem, and provide a few direct applications. While the proof of this topological version is notably simpler than its counterpart in the context of étale topology, its significance cannot be understated. This theorem holds paramount importance as the majority of formalisms within sheaf theory are built upon it.

1. The base change map

Let $f\colon X \to Y$ be a continuous map of spaces. We have the usual direct image functor

$$f_*\colon \mathsf{D}^+(X) \to \mathsf{D}^+(Y),$$

which is the right derived functor of the classical direct image functor $R^0 f_*$. Recall that for a sheaf $\mathcal{F}$ on X, and an open subset U of Y, we have

$$(R^0 f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

When Y reduces to a point, $R^m f_* \mathcal{F} = H^m(X; \mathcal{F})$ reduces to the usual sheaf cohomology of $\mathcal{F}$.

The following simple lemma gives a more concrete understanding of the direct image functor.

Lemma 1.1. *For any sheaf $\mathcal{F}$ on X, $R^m f_* \mathcal{F}$ is the sheaf on Y associated to the presheaf*

$$U \mapsto H^m(f^{-1}(U); \mathcal{F}).$$

(See Stacks Project, Tag 01E4.)

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^m f_* \mathcal{F}$ is computed as the m^{th} cohomology sheaf of the chain complex

$$f_* \mathcal{I}^0 \to f_* \mathcal{I}^1 \to \cdots.$$

which is the sheaf associated to the presehaf

$$V \longmapsto \frac{\operatorname{Ker}(f_* \mathcal{I}^i(V) \to f_* \mathcal{I}^{i+1}(V))}{\operatorname{Im}(f_* \mathcal{I}^{i-1}(V) \to f_* \mathcal{I}^i(V))},$$

the right hand side equals $H^i(f^{-1}(V), \mathcal{F})$. □

We also have the inverse image functor $f^*\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathrm{X})$. The direct image functor f_* is a right adjoint to f^*. If $\mathcal{G}$ is a sheaf on Y, then the stalk of $f^*\mathcal{G}$ at a point x of X equals the stalk of $\mathcal{G}$ at $f(x)$.

Let $\mathcal{F}$ be a sheaf on X. Then the stalk of $\mathrm{R}^m f_*\mathcal{F}$ at a point y is related to the cohomology $\mathrm{H}^m(f^{-1}(y);\mathcal{F})$ via a "base change" map, described below.

From Lemma 1.1, we see

$$(\mathrm{R}^m f_*\mathcal{F})_y = \operatorname*{colim}_{\mathrm{U}} \mathrm{H}^m(f^{-1}(\mathrm{U});\mathcal{F}),$$

and there are compatible restriction maps $\mathrm{H}^m(f^{-1}(\mathrm{U});\mathcal{F}) \to \mathrm{H}^m(f^{-1}(y);\mathcal{F}|_{f^{-1}(y)})$. Combining these we get a base change map

$$(\mathrm{R}^m f_*\mathcal{F})_y \to \mathrm{H}^m(f^{-1}(y);\mathcal{F}|_{f^{-1}(y)}). \tag{1.2}$$

In general, however, (1.2) is neither injective nor surjective.

More generally, for any cartesian diagram

$$\begin{array}{ccc} \mathrm{X}' & \xrightarrow{g'} & \mathrm{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Y}' & \xrightarrow{g} & \mathrm{Y} \end{array} \tag{1.3}$$

we can define a base change map

$$g^* f_*\mathcal{F} \to f'_* g'^*\mathcal{F} \tag{1.4}$$

for any object $\mathcal{F}$ of $\mathrm{D}^+(\mathrm{X})$. This map reduces to (1.2) when Y′ reduces to a point.

Instead of using the characterization of higher direct images from Lemma 1.1, one can also construct the base change map via the adjunctions:

$$g^* f_* \to f'_* f'^* g^* f_* = f'_* g'^* f^* f_* \to f'_* g'^*.$$

If g is a local homeomorphism (e.g., the inclusion map of an open set), then (1.4) is an isomorphism. In general, it is not an isomorphism.

2. Proper base change theorem: Statement

A condition that ensures the base change map (1.4) is an isomorphism is the *properness* of the map.

[1] Definition 2.1 (Proper map). A continuous map $f\colon \mathrm{X} \to \mathrm{Y}$ of spaces is called a *proper map*, if

(i) f is a *universally closed map*, that is, for any map $g\colon \mathrm{T} \to \mathrm{Y}$, the induced map $\mathrm{X} \times_{\mathrm{Y}} \mathrm{T} \to \mathrm{T}$ is a closed map;

(ii) f is a *separated map*, i.e., for every two points x_1, x_2 of X such that $f(x_1) = f(x_2)$, there exists an open neighborhood U_1 of x_1, an open neighborhood U_2 of x_2, such that $\mathrm{U}_1 \cap \mathrm{U}_2 = \emptyset$.

Example 2.2. Let Z be a closed subspace of a topological space X, then the inclusion map $\mathrm{Z} \hookrightarrow \mathrm{X}$ is proper. If X is a compact Hausdorff space, then $\mathrm{X} \to \{*\}$ is proper.

[2] Theorem 2.3 (Proper base change theorem). *In the situation of* (1.3)*, assume that f is proper. Then the base change map* (1.4) *is an isomorphism for any object $\mathcal{F}$ of* $\mathrm{D}^+(\mathrm{X})$. (See [SGA 4, $\mathrm{V}^{\mathrm{bis}}$, Théorème 4.1.1], or Stacks Project, `Tag 09V6`.)

Before proving the proper base change, let us mention that the properness hypothesis of Theorem 2.3 cannot be dropped. Here is an example.

Example 2.4. Consider $\mathrm{X} = \mathbf{C}^2 - \{0\}$, $\mathrm{Y} = \mathbf{C}$, and $f\colon \mathrm{X} \to \mathrm{Y}$ is defined by $f(x, y) = y$. In this case, f is not proper, and we claim that the conclusion of Theorem 2.3 does not hold for f.

For $y = 0$, we have

$$(\mathrm{R}^1 f_*\mathbf{Z})_0 = \operatorname*{colim}_{\epsilon \to 0} \mathrm{H}^1((\mathbf{C} \times \Delta_\epsilon) - \{(0,0)\};\mathbf{Z}),$$

where Δ_ϵ denotes a disk of radius ϵ around 0. Since the ambient space $(\mathbf{C} \times \Delta_\epsilon) - \{(0,0)\}$ is homotopy equivalent to $\mathbf{S}^3$, the right hand side is zero. On the other hand, $f^{-1}(0) = \mathbf{C}^*$, whose first cohomology is isomorphic to $\mathbf{Z}$.

Here is a very often used consequence of the proper base change theorem: in a one-parameter degeneration, the cohomology of the central fiber agrees with the cohomology of the total space over a sufficiently small disk.

EXAMPLE 2.5. Let $f\colon \mathrm{X} \to \Delta$ be a proper map, where Δ is a small disk in $\mathbf{C}$. Let $\Delta^* = \Delta - \{0\}$, and $\mathrm{X}^* = f^{-1}(\Delta^*)$. Assume that the topological type of $f^{-1}(\{|z| < \epsilon\}) \to \{|z| < \epsilon\}$ remains constant for $\epsilon \ll 1$. Then we have $\mathrm{H}^*(\mathrm{X}) \simeq \mathrm{H}^*(f^{-1}\{|z| < \epsilon\})$. By applying Theorem 2.3, we deduce that

$$\mathrm{H}^*(\mathrm{X}) \simeq \mathrm{H}^*(f^{-1}(0)).$$

The requirement of the constancy of the topological type is not unreasonable and is satisfied in various scenarios. For example, it occurs when $\mathrm{X} - f^{-1}(0)$ is a manifold, and f has no critical values in Δ away from 0. In such cases, it is possible to lift a radiant vector field on the disk to $\mathrm{X} - f^{-1}(0)$ and utilize a flow to obtain the desired homeomorphism. More generally, the requirement is met if X is a Whitney stratified subspace of a manifold, and f has no critical values, as defined by Goresky and MacPherson [GM88, Introduction, §1.2], away from 0. See loc. cit., SMT, Part A.

To end this section, we discuss an important consequence of the proper base change theorem, namely the base change theorem for the *extraordinary* direct image functor. We first give the definition.

DEFINITION 2.6. For any map $f\colon \mathrm{X} \to \mathrm{Y}$ between spaces, we define the functor $\mathrm{R}^0 f_!$ by setting

$$\mathrm{R}^0 f_! \mathcal{F}(\mathrm{U}) = \{s \in \mathcal{F}(f^{-1}(\mathrm{U})) : \operatorname{Supp} s \text{ is proper over } \mathrm{U}\}.$$

One checks that $\mathrm{R}^0 f_!$ is a left exact functor on plain sheaves of abelian groups, and we can therefore define its right derived functor $f_!\colon \mathrm{D}^+(\mathrm{X}) \to \mathrm{D}^+(\mathrm{Y})$. By definition, there is a natural transformation $f_! \to f_*$.

When Y reduces to a singleton, we shall write $f_!\mathcal{F}$ as $\mathrm{R}\Gamma_c(\mathrm{X}; \mathcal{F})$. Its cohomology are the *compactly supported cohomology* with coefficients in $\mathcal{F}$.

EXAMPLE 2.7. Here are two simple examples. When f is proper, e.g., when f is the inclusion map of a closed subspace, then $f_* = f_!$. When f is an open immersion, then $f_!$ is the usual extension by zero functor.

CONSTRUCTION 2.8 (Base change map for extraordinary direct image). Given the cartesian diagram (1.3), we constructed a base change morphism (1.4). Now we explain how it in turn induces a base change morphism for the extraordinary direct image

$$g^* f_! \mathcal{F} \to f'_! g'^* \mathcal{F}. \tag{2.9}$$

Let $\mathcal{F}$ be an injective sheaf, then for any open subset V of Y′, the left hand side is computed by the formula

$$g^* f_! \mathcal{F}(\mathrm{V}) = \operatorname*{colim}_{\mathrm{U} \supset f(\mathrm{V})} f_! \mathcal{F}(\mathrm{U}),$$

whereas the right hand side is computed by

$$f'_! g'^* \mathcal{F}(\mathrm{V}) = \{t \in g'^* \mathcal{F}(f'^{-1}(\mathrm{V})) : \operatorname{Supp}(t) \text{ is proper over } \mathrm{V}\}.$$

For any open subset U of Y such that $g(\mathrm{V}) \subset \mathrm{U}$, we have the following cartesian diagram

$$\begin{array}{ccc} f'^{-1}(\mathrm{V}) & \longrightarrow & f^{-1}(\mathrm{U}) \\ \downarrow & & \downarrow \\ \mathrm{V} & \longrightarrow & \mathrm{U}. \end{array}$$

This gives a pullback map $\mathcal{F}(f^{-1}(\mathrm{U})) \to (g'^*\mathcal{F})(f'^{-1}(\mathrm{V}))$. For any section s of $\mathcal{F}$ over $f^{-1}(\mathrm{U})$, the support of its inverse image g'^*s is contained in $\mathrm{U} \times_{\mathrm{V}} \operatorname{Supp}(s)$, by continuity. If $\operatorname{Supp}(s)$ is proper over U, it follows that $\mathrm{V} \times_{\mathrm{U}} \operatorname{Supp}(s)$, hence its closed subspace $\operatorname{Supp}(g'^*s)$ is proper over V. This gives a map $f_!\mathcal{F}(\mathrm{U}) \to f_!^* g'^* \mathcal{F}(\mathrm{V})$. Taking colimit gives the desired map (2.9).

LEMMA 2.10. *Let $j\colon \mathrm{U} \to \mathrm{X}$ be the inclusion map of an open subset of a space* X. *For a continuous map $g\colon \mathrm{X}' \to \mathrm{X}$, let $j'\colon \mathrm{U}' \to \mathrm{X}'$ be the pullback of j, and $g'\colon \mathrm{U}' \to \mathrm{U}$ the projection. Then for any $\mathcal{F} \in \mathrm{D}^+(\mathrm{U})$, the base change map $g^* j_! \mathcal{F} \to j'_! g'^* \mathcal{F}$ is an isomorphism.*

Proof. Checking stalks, we find both are the extension by zero of $g'^*\mathcal{F}$. □

[3] Theorem 2.11. *Given a cartesian diagram* (1.3)*, assume that* X *and* Y *are locally compact Hausdorff spaces. Then for any object* $\mathcal{F}$ *of* $\mathrm{D}^+(\mathrm{X})$*, the base change map* (2.9) *is an isomorphism.*

Proof. If f is an open immersion, we reduce to the simple Lemma 2.10 above. If f is proper, then we reduce to the usual proper base change theorem, Theorem 2.3. In both cases, we do not need any local compactness hypothesis. In the general case, the local compactness is used to employ Construction 2.12 below to write f as a composition $f_1 \circ f_2 \circ f_3$, with f_1 and f_3 proper, and f_2 an open immersion. That the base change map is an isomorphism follows immediately from some elementary manipulations. □

Construction 2.12. Let X be a locally compact Hausdorff space. Denote by $j\colon \mathrm{X} \to \overline{\mathrm{X}}$ the one-point compactification of X [Mun00, pp. 183–185]. Then any *separated* continuous map $f\colon \mathrm{X} \to \mathrm{Y}$ can be factored as a sequence of compositions:

$$\begin{array}{ccccc}
\mathrm{X} & \xhookrightarrow{\Gamma_f} & \mathrm{X}\times\mathrm{Y} & \xhookrightarrow{j\times \mathrm{Id}_\mathrm{Y}} & \overline{\mathrm{X}}\times\mathrm{Y} \\
\downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle \mathrm{pr}_2} \\
\mathrm{Y} & = & = & = & \mathrm{Y}
\end{array}$$

where Γ_f is the diagram embedding. Consequently, the functor $f_!$ can be decomposed as a composition $\mathrm{pr}_{2*} \circ (j \times \mathrm{Id}_\mathrm{Y})_! \circ \Gamma_{f*}$.

3. Proper base change theorem: Proof

Theorem 2.3 is equivalent to the following corollary; the corollary is obtained by taking Y' to be a point of Y, and the theorem is obtained by applying the corollary to f'.

Corollary 3.1. *Let* $f\colon \mathrm{X} \to \mathrm{Y}$ *be a proper map. Let* $\mathcal{F}$ *be a sheaf on* X*. Then for any point* y *of* Y*, the canonical map* (1.2)

$$(\mathrm{R}^i f_*\mathcal{F})_y \longrightarrow \mathrm{H}^i(f^{-1}(y); \mathcal{F})$$

is an isomorphism.

By definition, $(\mathrm{R}^i f_*\mathcal{F})_y = \operatorname{colim} \mathrm{H}^i(f^{-1}(\mathrm{U}); \mathcal{F})$, where U runs through all the open neighborhoods of y in Y. When f is a closed map, $f^{-1}(\mathrm{U})$ form a fundamental system of neighborhoods of $f^{-1}(y)$. Therefore, Corollary 3.1 is a consequence of the following lemma.

Lemma 3.2. *Let* X *be a topological space. Let* Z *be a compact subset of* X *such that any two distinct points of* Z *have neighborhoods in* X *that are disjoint. Then for any sheaf* $\mathcal{F}$ *on* X*, the natural map*

$$\operatorname{colim} \mathrm{H}^i(\mathrm{V}; \mathcal{F}) \longrightarrow \mathrm{H}^i(\mathrm{Z}; \mathcal{F}|_\mathrm{Z})$$

is an isomorphism, where the colimit is taken over all open subsets of X *containing* Z. (See [SGA 4, $\mathrm{V}^{\mathrm{bis}}$ 4.1.3], or [God58, II 4.11.1], or Stacks Project, `Tag 09V3`.)

We first prove the $i = 0$ case of the lemma. The injectivity is clear. To prove the surjectivity, we use the following lemma.

Lemma 3.3. *Under the hypothesis of Lemma 3.2, let* A *and* B *be two closed subsets of* Z *and* W *is an open neighborhood of* $\mathrm{A} \cap \mathrm{B}$ *in* X*. Then there exists open neighborhoods* U *and* V *of* A *and* B *in* X*, such that* $\mathrm{U} \cap \mathrm{V} \subset \mathrm{W}$.

Proof. We first treat the case when A is reduced to a point a. When $a \in \mathrm{B}$, then the lemma is trivial, since we can take $\mathrm{U} = \mathrm{V} = \mathrm{W}$. If $a \notin \mathrm{B}$, then W is an arbitrary open subset of X. For each $b \in \mathrm{B}$, we can find an open neighborhood U_a of a and V_b of b in X such that $\mathrm{U}_a \cap \mathrm{V}_b = \emptyset$. Since B is closed in a compact space Z, it is itself compact. Thus we can finitely many $\mathrm{V}_{b_1}, \ldots, \mathrm{V}_{b_r}$ such that $\mathrm{B} \subset \mathrm{V}_{b_1} \cup \cdots \cup \mathrm{V}_{b_r}$. Now we take $\mathrm{U} = \bigcap_{j=1}^r \mathrm{U}_{b_j}$, and $\mathrm{V} = \bigcup \mathrm{V}_{b_j}$, then $\mathrm{U} \cap \mathrm{V} = \emptyset$, which satisfies the requirement.

In the general case, for each $a \in \mathrm{A}$ and $b \in \mathrm{B}$, we can find open neighborhoods U_a and V_b of a and b in X, such that $\mathrm{U}_a \cap \mathrm{V}_b \subset \mathrm{W}$. Again by compactness we can find finitely many $\mathrm{U}_{a_1}, \ldots, \mathrm{U}_{a_s}$ that covers A. We take $\mathrm{U} = \bigcup_{j=1}^s \mathrm{U}_{a_j}$, and $\mathrm{V} = \bigcap_{j=1}^s \mathrm{V}_{b_j}$. □

3.4. Proof that the map in Lemma 3.2 is surjective for $i = 0$. For each section σ of $\mathcal{F}$ on Z, we can find an finite collection of open sets $U_1, \dots, U_r$ of X, such that σ extends to a section σ_i of $\mathcal{F}$ on U_i. Since Z is compact, we may also find an open covering $V_1, \dots, V_r$ of Z such that $\overline{V}_i \subset U_i \cap Z$.

Since $\sigma_i|_{U_i \cap U_j \cap Z} = \sigma_j|_{U_i \cap U_j \cap Z}$, we could find an open subset W_{ij} of X containing $\overline{V}_i \cap \overline{V}_j$ such that $\overline{V}_i \cap \overline{V}_j \subset W_{ij}$, and $\sigma_i|_{W_{ij}} = \sigma_j|_{W_{ij}}$. By Lemma 3.3, we can find open neighborhoods $\widetilde{V}_i$ of $\overline{V}_i$ in X such that $\widetilde{V}_i \cap \widetilde{V}_j \subset W_{ij}$. Thus $\{\sigma_i|_{\widetilde{V}_i}\}$ glue to a section $\widetilde{\sigma}$ of $\mathcal{F}$ over $\bigcup \widetilde{V}_i$. By construction $\widetilde{\sigma}|_Z = \sigma$. This proves the surjectivity, and finishes the proof of Lemma 3.2 for $i = 0$.

LEMMA 3.5. *Hypothesis as in Lemma 3.2, assume that $\mathcal{F}$ is a flasque sheaf. Then $\mathcal{F}|_Z$ is a mou sheaf.*

Recall that a sheaf $\mathcal{F}$ on a topological space X is said to be *flasque* (English: *flabby*), if for any open set U of X, the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective; $\mathcal{F}$ is said to be *mou* (English: *soft*), if for any closed subset Z of X, the map $H^0(X; \mathcal{F}) \to H^0(Z; \mathcal{F}|_Z)$ is surjective. Flasque sheaves are acyclic with respect to the global section functor; and mou sheaves are acyclic with respect to the global section functor if X is paracompact and Hausdorff [Gro57, 3.3.2, Corollaire].

PROOF OF LEMMA 3.5. Let A be a closed subset of Z. Any section of $\mathcal{F}$ on A extends to a neighborhood of A in X, thanks to the $i = 0$ case of Lemma 3.2 we just proved. Since $\mathcal{F}$ is assumed to be flasque, such a section extends to a section of $\mathcal{F}$ on X, and a fortiori to Z. □

END OF THE PROOF OF LEMMA 3.2. Let $\mathcal{J}^\bullet$ be a flasque resolution of $\mathcal{F}$. We have

$$\begin{aligned} \operatorname{colim} H^i(U; \mathcal{F}) &= \operatorname{colim} H^i(\mathcal{J}^\bullet(U)) && \text{by definition,} \\ &= H^i \operatorname{colim} \mathcal{J}^\bullet(U) && \text{by exactness of colimit,} \\ &= H^i \mathcal{J}^\bullet(Z) && \text{by 3.4,} \\ &= H^i(Z; \mathcal{F}|_Z) && \text{by 3.5.} \end{aligned}$$

This completes the proof of Lemma 3.2 and therefore the proper base change theorem. □

4. Some applications

In this section we provide two immediate applications of the proper base change theorem. The Vietoris–Begle theorem and the Künneth theorem.

4a. The Vietoris–Begle theorem. It is entirely possible to construct the framework of homology and cohomology theory for topological spaces using the language of sheaves, without the need to explicitly invoke the traditional singular theory. This is not a very wise choice, however, since one has to have the full arsenal of homological algebra handy.

However, if one remains steadfast in pursuing this path and successfully accomplishes it, the subsequent challenge lies in executing meaningful calculations. A prime example involves demonstrating the acyclicity of the closed interval with respect to the constant sheaf, and establishing the homotopy invariance of the cohomology of constant sheaves. We give these computations below.

DEFINITION 4.1. A topological space X is *acyclic* if for any abelian group E, the natural map $E \to R\Gamma(X; a_X^* E)$ is an isomorphism, where $a_X \colon X \to \{*\}$ is the canonical map.

PROPOSITION 4.2 (Vietoris–Begle). *Let $f \colon X \to Y$ be a proper map of topological spaces. Assume that for every $y \in Y$, the fiber $f^{-1}(y)$ is acyclic. Then for any sheaf $\mathcal{F}$ on Y, the natural map $\mathcal{F} \to f_* f^* \mathcal{F}$ is an isomorphism. Consequently, the natural morphisms $R\Gamma(Y; \mathcal{F}) \to R\Gamma(X; f^*\mathcal{F})$ and $R\Gamma_c(Y; \mathcal{F}) \to R\Gamma_c(X; f^*\mathcal{F})$ are isomorphisms.*

PROOF. In order to show $\mathcal{F} \to f_* f^* \mathcal{F}$ is an isomorphism, it suffices to check for every point y, $E := \mathcal{F}_y$ is isomorphic to $(f_* f^* \mathcal{F})_y$. By proper base change, Theorem 2.3, the right hand side is $R\Gamma(f^{-1}(y); E)$. The hypothesis ensures that the natural map $E \to R\Gamma(f^{-1}(y); E)$ is an isomorphism in the derived category, whence the proposition. □

The following proposition provides examples of spaces satisfying the hypothesis of the Vietoris–Begle theorem.

Proposition 4.3. *Let* I *be the closed interval* $[0,1]$. *Then for sheaf* $\mathcal{F}$ *on* I, $\mathrm{H}^i(\mathrm{I};\mathcal{F}) = 0$ *for* $i > 1$. *If moreover* $\mathcal{F}(\mathrm{I}) \to \mathcal{F}_t$ *is surjective for any* $t \in \mathrm{I}$, *then* $\mathrm{H}^1(\mathrm{I};\mathcal{F}) = 0$. (See [KS90, Proposition 2.7.1].)

Proof. Fix $j \geq 1$. Let $\mathrm{J} = \{t \in \mathrm{I} : \mathrm{H}^j(\mathrm{I};\mathcal{F}) \to \mathrm{H}^j([0,t];\mathcal{F}) = 0\}$. By Lemma 3.2, $\mathrm{H}^j([0,t];\mathcal{F}) = \operatorname{colim}_{t'>t} \mathrm{H}^j([0,t'];\mathcal{F})$, it follows that J is open. Let $t_0 = \sup \mathrm{J}$. We want to show that $t_0 \in \mathrm{J}$ as well. If either $j > 1$, or $j = 1$ but $\mathcal{F}(\mathrm{I}) \to \mathcal{F}_t$ is surjective for any t, then we have, by Mayer–Vietoris, for any $t \in \,]0,t_0[$,

$$\mathrm{H}^j([0,t_0];\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^j([0,t];\mathcal{F}) \oplus \mathrm{H}^j([t,t_0];\mathcal{F}).$$

By Lemma 3.2 again, we have $\operatorname{colim}_{t<t_0} \mathrm{H}^j([t,t_0];\mathcal{F}) = \mathrm{H}^j(\{t_0\};\mathcal{F}) = 0$, for any element ξ of $\mathrm{H}^j(\mathrm{I};\mathcal{F})$, there is t such that ξ is mapped to zero in $\mathrm{H}^j([t,t_0];\mathcal{F})$. Since $t < t_0$, ξ is mapped to zero in $\mathrm{H}^j([0,t];\mathcal{F})$ as well. Thus ξ is sent to zero in $\mathrm{H}^j([0,t_0];\mathcal{F})$. This concludes the proof. □

Proposition 4.4 (Generalized Vietoris–Begle). *Let* $f\colon \mathrm{X} \to \mathrm{Y}$ *be a continuous map of topological spaces. Assume that* X *is a union* $\mathrm{X} = \bigcup \mathrm{X}_n$, *such that*

(i) *for any* n, $f|_{\mathrm{X}_n}$ *is proper, and its fibers are acyclic,*
(ii) *we have* $\cdots \Subset \mathrm{X}_n \Subset \mathrm{X}_{n+1} \Subset \cdots$.

Then for any sheaf $\mathcal{F}$ *on* Y, *we have* $\mathcal{F} \simeq f_*f^*\mathcal{F}$. (See [KS90, Proposition 2.7.8].)

Proof. By the sheaf condition, we know that $f_*f^*\mathcal{F} = \lim f_{n*}f_n^*\mathcal{F}$. There is a spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{R}^i \lim \mathrm{R}^j f_{n*}f_n^*\mathcal{F} \Rightarrow \mathrm{R}^{i+j} f_*f^*\mathcal{F}$$

Since $\mathrm{X}_n \to \mathrm{Y}$ has acyclic fibers, by Vietoris–Begle (Proposition 4.2) and Mittag–Leffler, we have

$$\mathrm{E}_2^{i,j} = \begin{cases} \mathcal{F} & i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we know that $\mathrm{R}^i f_*f^*\mathcal{F} = 0$ for $i > 0$, and $\mathrm{R}^0 f_*f^*\mathcal{F} \simeq \mathcal{F}$. This completes the proof. □

Corollary 4.5. *Let* $\pi\colon \mathrm{E} \to \mathrm{B}$ *be a continuous map. Assume that* B *admits an open covering* $(\mathrm{U}_i)_{i\in\mathrm{I}}$, *such that the* U_i*-spaces* $\mathbf{R}^n \times \mathrm{U}_i$ *and* $\pi^{-1}(\mathrm{U}_i)$ *are isomorphic. Then* $\pi_*\mathbf{Z}_\mathrm{E} = \mathrm{R}^0\pi_*\mathbf{Z}_\mathrm{E}$ *is a rank one locally constant sheaf of abelian groups on* B.

4b. Projection formula. In sheaf theory we will make frequent use of projeciton formula. But this does not come for free. In this paragraph let us explain its content and proof.

Construction 4.6 (Projection formula). Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a continuous map of spaces. Let $\mathcal{F} \in \mathrm{D}^+(\mathrm{X})$ and $\mathcal{G} \in \mathrm{D}^+(\mathrm{Y})$. Then the natural map $f_*\mathcal{F} \otimes \mathcal{G}$ is well defined (essentially because the global dimension of $\mathbf{Z}$ is 1, so we do not have to worry about the bounded above issue), and have a natural isomorphism $f^*(f_*\mathcal{F} \otimes \mathcal{G}) \simeq f^*f_*\mathcal{F} \otimes f^*\mathcal{G}$. Applying the adjunction (f_*, f^*) gives a morphism $f^*(f_*\mathcal{F} \otimes \mathcal{G}) \to \mathcal{F} \otimes f^*\mathcal{G}$, and whence a morphism

$$f_*\mathcal{F} \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^*\mathcal{G}). \tag{4.7}$$

We shall refer to this morphism as the *projection morphism*. By a similar consideration as in Construction 2.8, we can get an extraordinary version of the projection morphism:

$$f_!\mathcal{F} \otimes \mathcal{G} \to f_!(\mathcal{F} \otimes f^*\mathcal{G}). \tag{4.8}$$

Example 4.9. The projection morphism (4.7) is not an isomorphism in general. To cook up an example, let X be a disjoint union of countably many points, Y a singleton, $\mathcal{F} = \mathbf{Z}_\mathrm{X}$, and $\mathcal{G} = \bigoplus \mathbf{Z}$ is the direct sum of countably many copies of $\mathbf{Z}$. Then the projection morphism reduces to the canonical homomorphism

$$\bigoplus_{\mathbf{N}} \Big(\prod_{\mathbf{N}} \mathbf{Z}\Big) \simeq \Big(\prod_{\mathbf{N}} \mathbf{Z}\Big) \otimes \bigoplus_{\mathbf{N}} \mathbf{Z} \to \prod_{\mathbf{N}} \Big(\bigoplus_{\mathbf{N}} \mathbf{Z}\Big).$$

This is not an isomorphism (see here).

Proposition 4.10. *Let* X *and* Y *be locally compact Hausdorff spaces. Then the extraordinary projection morphism* (4.8) *is an isomorphism.*

Proof. The proof reduces to proving the following assertion (the reduction step is left as an exercise). Let X be a compact space, $\mathcal{F}$ a plain sheaf on X, and M a torsion-free abelian group. Then we have a natural isomorphism:

$$\alpha\colon \Gamma(X;\mathcal{F}) \otimes_{\mathbf{Z}} M \xrightarrow{\sim} \Gamma(X;\mathcal{F} \otimes_{\mathbf{Z}} M), \tag{4.11}$$

cf. [KS90, Lemma 5.12]. The upshot of the proof is that for any $x \in X$, $\operatorname{colim} \mathcal{F}(U) \otimes M$ and $\operatorname{colim}(\mathcal{F} \otimes M)(U)$ are the same, since $\mathcal{F} \otimes M$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes M$. Therefore, if $\alpha(s) = 0$, we can find a *finite* open covering $(U_i)_{i\in I}$ of X such that $s = 0$ in $\mathcal{F}(U_i) \otimes M$. Since M is flat, and I is finite, the diagram

$$\mathcal{F}(X) \otimes M \longrightarrow \prod_i (\mathcal{F}(U_i) \otimes M) \rightrightarrows \prod_{i,j} (\mathcal{F}(U_i \cap U_j) \otimes M)$$

is still an equalizer diagram. From this we conclude that $s = 0$, i.e., α is injective. Proving the surjectivity is also left as an exercise. □

4c. The Künneth theorem. Consider a topological space X. Let $C_*(X)$ denote the chain complex of singular chains of X. Thus $C_m(X)$ is the free abelian group generated by the continuous maps from the geometric m-simplex Δ^m to X. The homology of this complex provides the singular homology of X.

Now, for two topological spaces X and Y, the Eilenberg–Zilber theorem ([Mun84, Theorem 59.2]) asserts that there is a natural chain homotopy equivalence

$$C_*(X) \otimes C_*(Y) \xrightarrow{\sim} C_*(X \times Y), \tag{4.12}$$

This forms the foundation for the familiar Künneth theorem through homological algebra. In other words, there is a split-exact sequence ([Mun84, Theorem 59.3]):

$$0 \to \bigoplus_{p+q=n} H_p(X;\mathbf{Z}) \otimes H_q(Y;\mathbf{Z}) \to H_n(X \times Y;\mathbf{Z}) \to \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(X;\mathbf{Z}), H_q(Y;\mathbf{Z})) \to 0. \tag{4.13}$$

When examining cohomology, there is some subtlety. Let $C^*(X) = \operatorname{Hom}(C_*(X), \mathbf{Z})$ be the singular cochain complex. By (4.12), we have $C^*(X \times Y) \simeq \operatorname{Hom}(C_*(X) \otimes C_*(Y), \mathbf{Z})$. Ideally, we would desire the right-hand side to be isomorphic to $C^*(X) \otimes C^*(Y)$. In fact, there does exist a natural map, called the *Künneth map*:

$$\operatorname{Hom}(C_*(X), \mathbf{Z}) \otimes \operatorname{Hom}(C_*(Y), \mathbf{Z}) \to \operatorname{Hom}(C_*(X) \otimes C_*(Y), \mathbf{Z}).$$

If $C_*(X)$ is a perfect complex, then the above Künneth map is an isomorphism. This is the case e.g., when Y has finitely generated cohomology in each degree. (See [Mun84, Theorems 60.3, 60.5].) However, this map is not an isomorphism in the general. Furthermore, in general it does not induce isomorphisms on the cohomology level either.

Example 4.14. Consider the case where $X = Y$ is a countably infinite set equipped with the discrete topology. In this scenario, we have $C_*(X) = \bigoplus_{\mathbf{N}} \mathbf{Z}$, while $C^*(X) = \prod_{\mathbf{N}} \mathbf{Z}$. The natural map:

$$\prod_{\mathbf{N}} \mathbf{Z} \otimes \prod_{\mathbf{N}} \mathbf{Z} \to \prod_{\mathbf{N}\times\mathbf{N}} \mathbf{Z}$$

is injective but not surjective.

This illustrates the non-isomorphism issue between the cochains of the tensor product and the tensor product of cochains. The solution is to use compactly supported cohomology (which in effect is a *homology theory*) in place of cohomology.

Proposition 4.15. *Let $f\colon X \to S$ and $g\colon Y \to T$ be two continuous maps between locally compact spaces. Then for $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$, we have a natural isomorphism*

$$(f \times g)_!(\mathcal{F} \boxtimes \mathcal{G}) \simeq f_!\mathcal{F} \boxtimes g_!\mathcal{G}.$$

Proof. By Theorem 2.11, it suffices to treat the case when S and T are singletons. In this case, let $\mathrm{pr}_Y\colon X \times Y \to Y$ be the natural projection, and similarly define pr_X. Then we have a cartesian diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_X} & X \\ {\scriptstyle \mathrm{pr}_Y}\downarrow & & \downarrow{\scriptstyle a_X} \\ Y & \xrightarrow{a_Y} & \{*\}. \end{array}$$

By Theorem 2.11 again, we have $a_{\mathrm{Y}}^{*}\mathrm{R}\Gamma_c(\mathrm{X};\mathcal{F}) \simeq \mathrm{pr}_{\mathrm{Y}!}(a_{\mathrm{X}}^{*}\mathcal{F})$. It follows from the projection formula, Proposition 4.10 that

$$\begin{aligned}\mathrm{R}\Gamma_c(\mathrm{X}\times\mathrm{Y};\mathcal{F}\boxtimes\mathcal{G}) &\simeq \mathrm{R}\Gamma_c(\mathrm{Y};\mathrm{pr}_{\mathrm{Y}!}(\mathcal{F}\boxtimes\mathcal{G}))\\ &\simeq \mathrm{R}\Gamma_c(\mathrm{Y};a_{\mathrm{Y}}^{*}\mathrm{R}\Gamma_c(\mathrm{X};\mathcal{F})\otimes\mathcal{G})\\ &\simeq \mathrm{R}\Gamma_c(\mathrm{X};\mathcal{F})\otimes\mathrm{R}\Gamma_c(\mathrm{Y};\mathcal{G}) \quad (\text{by } (4.11)).\end{aligned}$$

This completes the proof. □

Notes

1. For more characterizations of properness, see Stacks Project, Tag 005M. Let me only mention the following characterization which resembles the valuative criterion in algebraic geometry.

Let T be a topological space, and $\mathcal{U}$ an ultrafilter on T. Then we define a topological space $\mathrm{T}_{\mathcal{U}}$ as follows. As a set, it is $\mathrm{T}\sqcup\{\omega\}$; a subset of $\mathrm{T}_{\mathcal{U}}$ is open, if it is contained in T, or it is of the form $\mathrm{A}\sqcup\{\omega\}$, where $\mathrm{A}\in\mathcal{U}$.

Then a map of topological spaces $f\colon \mathrm{X}\to\mathrm{Y}$ is proper if and only if for any solid commutative diagram of continuous maps below,

$$\begin{array}{ccc}\mathrm{T} & \longrightarrow & \mathrm{X}\\ \downarrow & \nearrow & \downarrow f\\ \mathrm{T}_{\mathcal{U}} & \longrightarrow & \mathrm{Y}\end{array}$$

there always exists a unique dotted continuous map making the entire diagram commutative.

2. The proper base change theorem 2.3 can be generalized to the context of ∞-topoi. Concretely, if $f\colon \mathrm{X}\to\mathrm{Y}$ is a proper map of topological spaces, with X completely regular, then for any diagram

$$\begin{array}{ccccc}\mathcal{X}'' & \xrightarrow{u} & \mathcal{X}' & \longrightarrow & \mathrm{Shv}(\mathrm{X})\\ \downarrow f'' & & \downarrow f' & & \downarrow f\\ \mathcal{Y}'' & \xrightarrow{v} & \mathcal{Y}' & \longrightarrow & \mathrm{Shv}(\mathrm{Y})\end{array}$$

of ∞-topoi, in which the squares are cartesian, we have $v^{*}f'_{*} = f''_{*}u^{*}$ in the homotopy category. This is [Lur09, Theorem 7.3.1.16], and is termed "nonabelian proper base change theorem". Here Shv(X) is the ∞-topos of sheaves on X valued in the ∞-category of Kan complexes. Morphisms of ∞-topoi like f satisfying this property are called *proper morphisms* of ∞-topoi.

This nonabelian proper base change theorem can formally imply Theorem 2.3 when the spaces involved are all compact and Hausdorff. For a patient explanation, see the preprint of Peter J. Haine, "From nonabelian basechange to basechange with coefficients". However, it does not formally imply Theorem 2.3 in all situations, since the ∞-topos of sheaves on a fiber product is not necessarily isomorphic to the fiber product of ∞-topoi of sheaves.

3. A slightly more general version of Theorem 2.11 is given by Schnürer and Soergel [SS16]. We say a continuous map $f\colon \mathrm{X}\to\mathrm{Y}$ is *locally proper* if for any point x of X, and any open neighborhood U of x, there exists an open neighborhood V of y, a subset A of U, such that x is an interior point of A, $f(\mathrm{A})\subset\mathrm{V}$, and $f|_{\mathrm{A}}\colon \mathrm{A}\to\mathrm{V}$ is proper. It can be shown that being locally proper is stable under base change. The canonical map $a_{\mathrm{X}}\colon \mathrm{X}\to\{*\}$ is locally proper if and only if X is locally compact.

THEOREM. *Let $f\colon \mathrm{X}\to\mathrm{Y}$ be a locally proper and separated continuous map. Then for any cartesian diagram*

$$\begin{array}{ccc}\mathrm{X}' & \xrightarrow{g'} & \mathrm{X}\\ \downarrow f' & & \downarrow f\\ \mathrm{Y}' & \xrightarrow{g} & \mathrm{Y},\end{array}$$

we have $g^{*}f_{!} \xrightarrow{\sim} f'_{!}g'^{*}$.

The formulation introduced by Schnürer and Soergel is probably more preferable than the classical formulation, which does not impose constraints on f. In contrast, Schnürer and Soergel's formulation underscores the point that the base change theorem is a relative property that applies to maps, rather than an absolute property dependent on spaces.

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