Decomposition theorem for smooth proper morphisms

Contents

The Decomposition Theorem of Beilinson, Bernstein, and Deligne (and Gabber) is a result concerning a specific property of proper algebraic mappings between algebraic varieties. In order to state this theorem, the languages of derived categories, t-structures, and perverse sheaves have to be used. However, even with a substantial grasp of these concepts, one may still not be able to comprehend the significance of this theorem. The objective of this lecture is to provide the statement and proof of the Decomposition Theorem when the mapping is smooth, which hopefully can offer some insight and motivation. [1]

1. Blanchard's theorem

Let $f\colon \mathrm{X} \to \mathrm{B}$ be a map between nice topological spaces, such as geometric realizations of finite dimensional locally finite simplicial complexes. Assume that B has an open covering $\mathfrak{U} = (\mathrm{U}_i)_{i\in \mathrm{I}}$ such that as a space over U_i, $f^{-1}\mathrm{U}_i \to \mathrm{U}_i$ is isomorphic to $\mathrm{pr}_1\colon \mathrm{U}_i \times \mathrm{F} \to \mathrm{U}_i$, for some topological space F. We say f is a locally topologically trivial fibration. Given such a map f, Leray designed a method to compute the cohomology of X based on the knowledge of the cohomology of B and F. This is the Leray spectral sequence:

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{B}; \mathrm{R}^q f_* \mathbf{Q}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{X}; \mathbf{Q}).$$

The second page of the Leray spectral sequence is rather explicit: $\mathrm{R}^q f_* \mathbf{Q}$ captures the information of the cohomology of the fiber F, and how the fundamental group of B acts on $\mathrm{H}^q(\mathrm{F}; \mathbf{Q})$. The space $\mathrm{H}^p(\mathrm{B}; \mathrm{R}^q f_* \mathbf{Q})$, if computed using the Čech method, reflects how the cohomology of $f^{-1}(b)$ undergoes variations across B. This spectral sequence serves as a comprehensive summary of the map f. However, this second page is rarely the last page.

Example 1.1. Consider the Hopf fibration $\pi\colon \mathbf{S}^3 \to \mathbf{S}^2$, with $\mathrm{F} = \mathbf{S}^1$. Since the base is simply connected, $\mathrm{R}^1\pi_*\mathbf{Q}_{\mathbf{S}^3} = \mathrm{R}^0\pi_*\mathbf{Q}_{\mathbf{S}^3} = \mathbf{Q}_\mathrm{B}$. If the Leray spectral sequence were to degenerate at E_2, we would conclude that "Künneth formula" would hold for π. But this is absurd, as $\mathrm{H}^1(\mathbf{S}^3) = 0$.

Example 1.2. Even within more rigid branches of geometry like complex analytic geometry, hoping for the degeneration of the Leray spectral sequence at E_2 in general remains unrealistic. Here is an example akin to the complex geometric version of the Hopf fibration (the complex manifold X below is termed *Hopf surface*). Consider the total space U of the tautological line bundle over $\mathbf{P}^1$ with its zero section removed. Then U is a $\mathbf{C}^*$-bundle over $\mathbf{P}^1$. The group $\mathbf{Z}$ acts on U by scaling the fibers of $\mathrm{U} \to \mathbf{P}^1$ by a factor q (where q is any number such that $|q| < 1$). This action is free and without fixed points. The quotient space X of U under the $\mathbf{Z}$-action is a compact complex surface, and the projection $\mathrm{U} \to \mathbf{P}^1$ descends to a proper holomorphic map $f\colon \mathrm{X} \to \mathbf{P}^1$. It is notable that the fibers of f are all isomorphic to the elliptic curve $\mathbf{C}^*/q^{\mathbf{Z}}$, which is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$. A scrutiny (left to the reader) reveals that the Leray spectral sequence does *not* experience E_2-degeneration in this scenario.

The above discussion makes the following theorem of Blanchard looks very strong:

THEOREM 1.3 (Blanchard [2]). *If $f\colon \mathrm{X} \to \mathrm{Y}$ is a proper submersion between complex manifolds. Assume that* X *is Kähler. (In particular all the fibers are compact Kähler manifolds.) Then the Leray spectral sequence degenerates at* E_2.

Thus, in Kähler geometry, the affair appears considerably more manageable. The difference between the Kähler category and complex analytic category which facilitates the validity of Blanchard's theorem is a robust constraint on the cohomology of compact Kähler manifolds, the Lefschetz decomposition, which is a consequence of the hard Lefschetz theorem.

THEOREM 1.4 (Hard Lefschetz theorem). *Let* M *be a compact Kähler manifold of pure dimension n. Let ω be Kähler class of* M*. Then for $i = 1, 2, \dots, n$, the map*

$$\mathrm{H}^{n-i}(\mathrm{M}; \mathbf{C}) \xrightarrow{\eta \mapsto \eta \smile [\omega]^i} \mathrm{H}^{n+i}(\mathrm{M}; \mathbf{C})$$

is an isomorphism.

To state the Lefschetz decomposition, we need the following definition.

DEFINITION 1.5. Let M be a compact Kähler manifold of dimension n, and ω a Kähler class. Let us denote the map

$$\eta \mapsto \eta \smile [\omega]\colon \mathrm{H}^i(\mathrm{M}) \to \mathrm{H}^{i+2}(\mathrm{M})$$

by L. Then for $0 \le i \le n$, we define i^{th} *primitive cohomology* of M as

$$\mathrm{H}^i(\mathrm{M})_{\text{prim}} = \mathrm{Ker}(\mathrm{L}^{n-i}\colon \mathrm{H}^i(\mathrm{M}) \to \mathrm{H}^{n+i+2}(\mathrm{M})).$$

THEOREM 1.6 (Lefschetz decomposition). *For $i = 0, 1, \dots, n$, we have*

$$\mathrm{H}^i(\mathrm{M}) \simeq \mathrm{H}^i(\mathrm{M})_{\text{prim}} \oplus \mathrm{L}\mathrm{H}^{i-2}(\mathrm{M})_{\text{prim}} \oplus \mathrm{L}^2\mathrm{H}^{i-4}(\mathrm{M})_{\text{prim}} \oplus \cdots,$$

and

$$\mathrm{H}^{2n-i}(\mathrm{M}) \simeq \mathrm{L}^{n-i}\mathrm{H}^i(\mathrm{M})_{\text{prim}} \oplus \mathrm{L}^{n-i+1}\mathrm{H}^{i-2}(\mathrm{M})_{\text{prim}} \oplus \mathrm{L}^{n-i+2}\mathrm{H}^{i-4}(\mathrm{M})_{\text{prim}} \oplus \cdots.$$

[2] For the proofs of Theorem 1.4 and Theorem 1.6, one may consult [18, Chapitre IV, no.6 (p. 75)] or the more recent [17, Section 6.2.3].

PROOF OF THEOREM 1.3. To begin with, we comment that it suffices to prove the theorem with coefficients assumed to be $\mathbf{C}$ instead of $\mathbf{Q}$. This is because showing the differentials d_k are zero is equivalent to showing it to be zero after extending the field of scalars.

Since X is Kähler, any Kähler class ω of X restricts to a Kähler class on each fiber $f^{-1}(y)$. The Kähler class ω induces a map of direct images

$$u\colon f_*\mathbf{C}_\mathrm{X} \to f_*\mathbf{C}_\mathrm{X}[2].$$

which further induces $\mathrm{L}\colon \mathrm{R}^i f_*\mathbf{C}_\mathrm{X} \to \mathrm{R}^{i+2} f_*\mathbf{C}_\mathrm{X}$, and

$$\mathrm{L}^i\colon \mathrm{R}^{n-i} f_*\mathbf{C}_\mathrm{X} \to \mathrm{R}^{n+i} f_*\mathbf{C}_\mathrm{X}. \tag{1.7}$$

Since f is proper, the proper base change theorem implies that the above map, when restricted to y, is none other than the map L appeared in the usual hard Lefschetz theorem. Since being an isomorphism is a stalkwise property, we conclude that the map (1.7) is an isomorphism.

The relative version of the hard Lefschetz then gives a Lefschetz decomposition by the same linear algebra:

$$\begin{cases} \oplus_{k\ge 0} u^k: & \bigoplus_{k\ge 0}(\mathrm{R}^{n-i-2k} f_*\mathbf{C}_\mathrm{X})_{\text{prim}} \xrightarrow{\sim} \mathrm{R}^{n-i} f_*\mathbf{C}_\mathrm{X}, \\ \oplus_{k\ge 0} u^{i+k}: & \bigoplus_{k\ge 0}(\mathrm{R}^{n-i-2k} f_*\mathbf{C}_\mathrm{X})_{\text{prim}} \xrightarrow{\sim} \mathrm{R}^{n+i} f_*\mathbf{C}_\mathrm{X}. \end{cases} \tag{1.8}$$

Since u is defined on $f_*\mathbf{C}_X$, it induces a map of the Leray spectral sequence

$$u\colon E_r^{p,q} \to E_r^{p,q+2},$$

commuting with the differentials. One the level of E_2, we have $E_2^{p,q} = H^p(Y, R^q f_* \mathbf{C}_X)$, and u induces following commutative diagram:

$$\begin{array}{ccc} (E_2^{p,n-i})_{\text{prim}} & \xrightarrow{d_2} & E_2^{p+2,n-i-2+1} \\ \downarrow{\scriptstyle u^{i+1}} & & \downarrow{\scriptstyle u^{i+1}} \\ E_2^{p,n+i+2} & \xrightarrow{d_2} & E_2^{p+2,n+i-2+3}. \end{array}$$

The left vertical arrow is zero by the definition of primitive cohomology; the right vertical arrow is an isomorphism. This forces $d_2 = 0$. The similar argument works for all $r \geq 2$ as long as one proceed inductively. □

2. Splittings in derived category

A decade after Blanchard proved his theorem, Deligne [6] also noticed the degeneration of the Leray spectral sequence for a proper submersion $f\colon X \to Y$ between Kähler manifolds. Meanwhile, Deligne observed that $f_*\mathbf{Q}_X$ has very simple structure in the bounded derived category of Y.

An object in the derived category $D^b(Y)$ can be represented by a chain complex of sheaves $\mathcal{F}^\bullet$, and there can be multiple such representations. If two complexes of sheaves, $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$, become isomorphic objects in the derived category, then there exist a family of isomorphisms between cohomology sheaves: $\alpha^m\colon \mathcal{H}^m(\mathcal{F}^\bullet) \approx \mathcal{H}^m(\mathcal{G}^\bullet)$. However, having isomorphic cohomology sheaves is not sufficient for two complexes of sheaves to be isomorphic in the derived category. To establish an isomorphism in the derived category, a stronger condition is required. This entails introducing a third complex $\mathcal{T}^\bullet$, which admits chain maps $\mathcal{T}^\bullet \to \mathcal{F}^\bullet$ and $\mathcal{T}^\bullet \to \mathcal{G}^\bullet$; and these chain maps induce isomorphisms on cohomology sheaves.

For instance, the cohomology sheaves of $\mathcal{F}^\bullet$ are always isomorphic to the cohomology sheaves of the following complex with vanishing differentials:

$$\bigoplus_{m\in\mathbf{Z}} \mathcal{H}^m(\mathcal{F}^\bullet)[-m] : \qquad \cdots \to \mathcal{H}^{-1}(\mathcal{F}^\bullet) \xrightarrow{0} \mathcal{H}^0(\mathcal{F}^\bullet) \xrightarrow{0} \mathcal{H}^1(\mathcal{F}^\bullet) \to \cdots . \tag{2.1}$$

But in most cases, an object $\mathcal{F}^\bullet$ in the derived category will not be isomorphic to a complex with identical zero differentials. This situation is analogous to homotopy theory, where a homotopy type is determined by its Postnikov tower, a series of fibrations, whose fibers are all Eilenberg–Mac Lane spaces, but this tower is generally not a product of Eilenberg–Mac Lane spaces.

Returning to algebra, if $\mathcal{F}$ is isomorphic to (2.1) in the derived category, then the spectral sequence

$$E_2^{p,q} = H^p(Y; \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(Y; \mathcal{F}) \tag{2.2}$$

will degenerate at E_2. Of course, expecting this to occur all the time is unreasonable. This abstract reasoning clarifies why $\mathcal{F} \simeq \bigoplus \mathcal{H}^i(\mathcal{F})[-i]$ cannot be expected to hold universally.

The following simple example illustrates that even when the spectral sequence (2.2) degenerates at E_2, $\mathcal{F}$ still might not be isomorphic to (2.1) in the derived category. [3]

EXAMPLE 2.3. Consider the inclusion $j\colon \mathbf{G}_m \to \mathbf{A}^1$. Let $\mathcal{F} = j_*\mathbf{Z}_{\mathbf{G}_m}$. Then $\mathcal{H}^0(\mathcal{F}) = \mathbf{Z}_{\mathbf{A}^1}$, and $\mathcal{H}^1(\mathcal{F}) = \iota_*\mathbf{Z}$, where $\iota\colon \{0\} \to \mathbf{A}^1$ in the inclusion map of the origin. I claim $\mathcal{F}$ is not isomorphic to the complex $\mathbf{Z}_{\mathbf{A}^1} \xrightarrow{0} \iota_*\mathbf{Z}$. In fact, any morphism from $\iota_*\mathbf{Z}[-1]$ into $\mathcal{F}$ is zero: by adjunction,

$$\mathrm{Hom}_{D(\mathbf{A}^1)}(\iota_*\mathbf{Z}[-1], j_*\mathbf{Z}_{\mathbf{G}_m}) = \mathrm{Hom}_{D(\mathbf{G}_m)}(j^*\iota_*\mathbf{Z}[-1], \mathbf{Z}_{\mathbf{G}_m}),$$

but $j^*\iota_*\mathbf{Z}[-1]$ is the zero object of $D(\mathbf{G}_m)$.

Now we can state Deligne's result.

[4] THEOREM 2.4 (Deligne [6]). *Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a proper submersion between complex manifolds, whose fibers all have complex dimension n. Assume that* X *is a Kähler manifold. Then in the derived category* $\mathrm{D}^b(\mathrm{Y})$, $f_*\mathbf{Q}_{\mathrm{X}}$ *is isomorphic to*

$$\bigoplus_{m=0}^{2n} \mathrm{R}^m f_*\mathbf{Q}_{\mathrm{X}}[-m]: \qquad \mathrm{R}^0 f_*\mathbf{Q}_{\mathrm{X}} \xrightarrow{0} \mathrm{R}^1 f_*\mathbf{Q}_{\mathrm{X}} \xrightarrow{0} \cdots \xrightarrow{0} \mathrm{R}^{2n} f_*\mathbf{Q}_{\mathrm{X}}.$$

We will postpone the proof of the theorem to Section 3. Before delving into the proof, let's explore two relatively straightforward examples of the splitting phenomenon. The first example is the Leray–Hirsch theorem. Although Deligne's theorem is distinct from a Leray–Hirsch phenomenon, the Leray–Hirsch theorem plays a noteworthy role in proving the "Relative Hard Lefschetz Theorem" for pure perverse sheaves in [1].

THEOREM 2.5 (Leray–Hirsch). *Let $\pi\colon \mathrm{E} \to \mathrm{B}$ be a fiber bundle between manifolds. Assume that there exists a finite collection of homogeneous cohomology classes $\{\eta_1, \ldots, \eta_r\} \subset \mathrm{H}^*(\mathrm{E})$ satisfying the following condition:*

for any $y \in \mathrm{B}$, the restriction of $\{\eta_1, \ldots, \eta_r\}$ to $\mathrm{H}^*(\pi^{-1}(y))$ form a basis of $\mathrm{H}^*(\pi^{-1}(y))$.

Then the map

$$\bigoplus_{i=1}^{r} \mathrm{H}^{m-\deg \eta_i}(\mathrm{B}) \to \mathrm{H}^m(\mathrm{E}), \quad (\gamma_1, \ldots, \gamma_r) \mapsto \sum_{i=1}^{r} \gamma_i \smile \eta_i$$

is an isomorphism. (Cf. [17, §8.1.3]; [10, Theorem 4D.1]; [4, Theorem 5.11].)

PROOF. Each η_i defines a morphism $f^*\mathbf{Q}_{\mathrm{B}} = \mathbf{Q}_{\mathrm{E}} \to \mathbf{Q}_{\mathrm{E}}[\deg \eta_i]$, which by adjunction gives

$$\mathbf{Q}_{\mathrm{B}} \xrightarrow{\cdot \eta_i} f_*\mathbf{Q}_{\mathrm{E}}[\deg \eta_i].$$

Taking direct sum yields an arrow in $\mathrm{D}^b(\mathrm{B})$:

$$\bigoplus_{i=1}^{r} \mathbf{Q}_{\mathrm{B}}[-\deg \eta_i] \to f_*\mathbf{Q}_{\mathrm{E}}. \tag{2.6}$$

We claim that this arrow is an isomorphism in the derived category. The problem being local, we may check it locally. Whence we may assume that f is trivializable. In this case, that the displayed arrow is an isomorphism reduces to the Künneth theorem. □

REMARK 2.7. Under the hypothesis of Theorem 2.5, the direct image sheaf $\mathrm{R}^m f_*\mathbf{Q}_{\mathrm{E}}$ is a *constant sheaf* on B, and the following map

$$\bigoplus_{\deg \eta_i = m} \mathbf{Q}_{\mathrm{B}} \to \mathrm{R}^m f_*\mathbf{Q}_{\mathrm{E}}, \quad \sum a_i \mapsto \sum a_i \eta_i$$

is an isomorphism, as can be checked stalk-by-stalk. Thus the isomorphism (2.6) can be rewritten as $f_*\mathbf{Q}_{\mathrm{E}} \simeq \bigoplus \mathrm{R}^m f_*\mathbf{Q}_{\mathrm{E}}[-m]$.

EXAMPLE 2.8 (Cohomology of projective bundle). Let M be a differentiable manifold of dimension n. Let $\mathcal{E}$ be a locally free $\mathcal{C}^{\infty}_{\mathrm{M},\mathbf{C}}$-Module of constant rank $r+1$ on M. Let $\pi\colon \mathbf{P}\mathcal{E} \to \mathrm{M}$ be the projective bundle of $\mathcal{E}$, which is a differentiable manifold of dimension $n + r$.

On the $\mathbf{P}\mathcal{E}$, we have a short exact squence of locally free $\mathcal{C}^{\infty}_{\mathbf{P}\mathcal{E},\mathbf{C}}$-Modules

$$0 \to \mathcal{O}_\pi(-1) \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathcal{O}_\pi(-1)$ is an invertible Module on $\mathbf{P}\mathcal{E}$, whose restriction to each fiber $\pi^{-1}(x)$ gives the tautological bundle of the complex projective space $\mathbf{P}^r$

Let η be the Euler class of $\mathcal{O}_\pi(1) = \mathscr{H}om_{\mathcal{O}_{\mathbf{P}\mathcal{E}}}(\mathcal{O}_\pi(-1), \mathcal{O}_{\mathbf{P}\mathcal{E}})$. By the naturality of the Euler class, we see $\eta|_{\pi^{-1}(x)}$ is the Euler class of the dual of the tautological bundle of $\pi^{-1}(x) = \mathbf{P}^r$. Hence, the powers η^i, $i = 0, 1, \ldots, r$, after restricting to a fiber $\pi^{-1}(x)$, form a basis of $\mathrm{H}^*(\pi^{-1}(x))$. Theorem 2.5 implies that

$$\mathrm{H}^m(\mathbf{P}\mathcal{E}) \simeq \mathrm{H}^m(\mathrm{M}) \oplus \mathrm{H}^{m-2}(\mathrm{M}) \cdot \eta \oplus \mathrm{H}^{m-4}(\mathrm{M}) \cdot \eta^2 \oplus \cdots.$$

Next, let us show the splitting phenomenon also holds for the blowing up of a complex manifold along a closed submanifold. Let M be a complex manifold of pure dimension n. Let Z be a closed submanifold of codimension r. Let

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{i'} & \widetilde{\mathrm{M}} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{M} \end{array}$$

be the blowup square of M along Z. See [9, p. 602] or [17, §3.3.3] for the construction.

2.9. Here are some basic properties of the morphism π.

(1) $\widetilde{\mathrm{M}}$ is a complex manifold, and π is a proper holomorphic map.
(2) $\mathrm{E} = \pi^{-1}(\mathrm{Z})$ is a divisor of $\widetilde{\mathrm{M}}$, and E is isomorphic to the projectivized normal bundle of Z, i.e., $\mathrm{E} \simeq \mathbf{P}\mathrm{N}_{\mathrm{Z}/\mathrm{M}}$. Thus $\mathrm{E} \to \mathrm{Z}$ is a projective bundle of fiber dimension $r-1$.
(3) $\pi|_{\widetilde{\mathrm{M}}-\mathrm{E}}$ is an isomorphism between $\widetilde{\mathrm{M}} - \mathrm{E} = \pi^{-1}(\mathrm{M}-\mathrm{Z})$ and $\mathrm{M}-\mathrm{Z}$. In other words, π is a birational holomorphic map.

PROPOSITION 2.10. *In the situation above, we have*

$$\begin{aligned} \pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}} &\simeq \bigoplus_{i=0}^{r-1} \mathrm{R}^i\pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}}[-2i] \\ &\simeq \mathbf{Q}_{\mathrm{M}} \oplus \mathbf{Q}_{\mathrm{Z}}[-2] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r+2]. \end{aligned}$$

(See [17, §7.3.3].)

PROOF. Let $i\colon \mathrm{Z} \to \mathrm{M}$ be the inclusion of Z into M. Let $j\colon \mathrm{M}-\mathrm{Z} \to \mathrm{M}$ be the inclusion of the complement of Z. Consider the distinguished triangle

$$i_*i^!(\pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}}) \to \pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}} \to j_*j^*(\pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}}) \xrightarrow[\text{proper base change}]{2.9(1,3)} j_*\mathbf{Q}_{\mathrm{M}-\mathrm{Z}}.$$

By proper base change, we have $i^!\pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}} = \pi'_*i'^!\mathbf{Q}_{\widetilde{\mathrm{M}}}$. Since E is a smooth Cartier divisor in $\widetilde{\mathrm{M}}$, we have $i'^!\mathbf{Q}_{\widetilde{\mathrm{M}}} \simeq \mathbf{Q}_{\mathrm{E}}[-2]$. By 2.9(2), we may apply Example 2.8, and obtain $\pi'_*i'^!\mathbf{Q}_{\widetilde{\mathrm{M}}} \simeq \mathbf{Q}_{\mathrm{Z}}[-2] \oplus \mathbf{Q}_{\mathrm{Z}}[-4] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r]$. We then have a commutative diagram

$$\begin{array}{ccccc} i_*\mathbf{Q}_{\mathrm{Z}}[-2r] & \longrightarrow & \mathbf{Q}_{\mathrm{M}} & \longrightarrow & j_*\mathbf{Q}_{\mathrm{M}-\mathrm{Z}} \\ \downarrow & & \downarrow & & \| \\ i_*(\mathbf{Q}_{\mathrm{Z}}[-2] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r]) & \longrightarrow & \pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}} & \longrightarrow & j_*\mathbf{Q}_{\mathrm{M}-\mathrm{Z}} \end{array}$$

whose rows are distinguished triangles. This diagram extends to the following diagram in which each row and column is a distinguished triangle:

$$\begin{array}{ccccc} i_*\mathbf{Q}_{\mathrm{Z}}[-2r] & \longrightarrow & \mathbf{Q}_{\mathrm{M}} & \longrightarrow & j_*\mathbf{Q}_{\mathrm{M}-\mathrm{Z}} \\ \downarrow & & \downarrow & & \| \\ i_*(\mathbf{Q}_{\mathrm{Z}}[-2] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r]) & \longrightarrow & \pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}} & \longrightarrow & j_*\mathbf{Q}_{\mathrm{M}-\mathrm{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ i_*(\mathbf{Q}_{\mathrm{Z}}[-2] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r+2]) & = & i_*(\mathbf{Q}_{\mathrm{Z}}[-2] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r+2]) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ (i_*\mathbf{Q}_{\mathrm{Z}}[-2r])[1] & \longrightarrow & \mathbf{Q}_{\mathrm{M}}[1] & \longrightarrow & j_*\mathbf{Q}_{\mathrm{M}-\mathrm{Z}}[1] \end{array}$$

We leave it to the reader to verify that the left vertical triangle is a direct sum. It follows that the bottom middle arrow is zero, and it follows that we have $\pi_*\mathbf{Q}_{\widetilde{\mathrm{M}}} \simeq \mathbf{Q}_{\mathrm{M}} \oplus \mathbf{Q}_{\mathrm{Z}}[-2] \oplus \cdots \oplus \mathbf{Q}_{\mathrm{Z}}[-2r+2]$. □

3. Proof of Theorem 2.4

LEMMA 3.1. *Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a locally topologically trivial fibration between manifolds. Let $y \in \mathrm{Y}$ be a point. Assume that the Leray spectral sequence degenerates at E_2, then the image of the map*

$$\mathrm{H}^i(\mathrm{X}; \mathbf{Q}) \to \mathrm{H}^i(f^{-1}(y); \mathbf{Q})$$

equals the monodromy invariant part $\mathrm{H}^i(f^{-1}(y))^{\pi_1(\mathrm{Y},y)} = \mathrm{H}^0(\mathrm{Y}; \mathrm{R}^i f_* \mathbf{Q}_\mathrm{X})$.

PROOF. This lemma is easy. The map $\mathrm{H}^i(\mathrm{X}; \mathbf{Q}) \to \mathrm{H}^0(\mathrm{Y}; \mathrm{R}^i f_* \mathbf{Q}_\mathrm{X})$ is the edge map of the E_2-page of the spectral sequence. It is surjective because of the degeneration (Theorem 1.3). By unwinding the definitions, one sees that the sheaf cohomology $\mathrm{H}^0(\mathrm{Y}; \mathrm{R}^i f_* \mathbf{Q}_\mathrm{X})$ can be described by the fixed part of the action of $\pi_1(\mathrm{Y}, y)$ on $\mathrm{H}^i(\mathrm{Y}; \mathbf{Q})$. □

LEMMA 3.2. *Let $\mathcal{A}$ be an abelian category. Let K be an object in $\mathrm{D}^b(\mathcal{A})$. Suppose that there exist arrows $\pi_j\colon \mathrm{K} \to \mathrm{K}$ such that $\mathrm{H}^i(\pi_j) = \delta_{ij}$. Then $\mathrm{K} \simeq \bigoplus_j \mathrm{H}^j(\mathrm{K})[-j]$ in $\mathrm{D}^b(\mathcal{A})$.* (See [6, Théorème 1.11].)

PROOF. We prove by induction on the the length of K. When K has length 1, K is the shift of an object of $\mathcal{A}$. In this case, the lemma is trivially true. In the general situation, assume without loss of generality that $\mathrm{H}^i(\mathrm{K}) \neq 0$ only for $i = 0, -1, \ldots, -\mathrm{N}$. Then K fits into a distinguished triangle

$$\tau_{\leq -1}\mathrm{K} \to \mathrm{K} \to \mathrm{H}^0(\mathrm{K}) \xrightarrow{\alpha} (\tau_{\leq -1}\mathrm{K})[1].$$

To show that $\mathrm{H}^0(\mathrm{K})$ is a direct factor of K, we shall show $\alpha = 0$. Since we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(\mathrm{K}) & \xrightarrow{\alpha} & (\tau_{\leq -1}\mathrm{K})[1] \\ {\scriptstyle \mathrm{H}^0(\pi_0)}\downarrow & & \downarrow{\scriptstyle \tau_{\leq -1}(\pi_0)[1]} \\ \mathrm{H}^0(\mathrm{K}) & \xrightarrow{\alpha} & (\tau_{\leq -1}\mathrm{K})[1], \end{array}$$

and since $\mathrm{H}^0(\pi_0)$ is by hypothesis an isomorphism, it suffices to show the right vertical arrow is zero. As the endomorphisms π_j for $j \leq -1$ induce endomorphisms of $\tau_{\leq -1}\mathrm{K}$, which by induction splits $\tau_{\leq -1}\mathrm{K}$: $\tau_{\leq -1}\mathrm{K} \approx \bigoplus_{j=1}^{\mathrm{N}-1} \mathrm{H}^{-j}(\mathrm{K})[j]$. Thus, proving $\tau_{\leq -1}(\pi_0)[1]$ being zero reduces to $\mathrm{H}^j(\pi_0) = 0$ for $j < 0$, which is our hypothesis. Therefore $\alpha = 0$, and we have $\mathrm{K} \approx \mathrm{H}^0(\mathrm{K}) \oplus \tau_{\leq -1}\mathrm{K} \approx \bigoplus_{j=0}^{\mathrm{N}} \mathrm{H}^{-j}(\mathrm{K})[j]$, as desired. □

REMARK 3.3. As a warning, we remark that if $f\colon \mathrm{K} \to \mathrm{L}$ is a map in the derived category and if $\mathrm{H}^j(f)$ are zero, we cannot deduce that f is the zero map in the derived category. For example, any map between the cohomology groups of the complexes of abelian groups $(\mathbf{Z}/2\mathbf{Z})[0]$ and $(\mathbf{Z}/2\mathbf{Z})[1]$ are zero, but there is a nontrivial extension

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

which corresponds to a nontrivial element in $\mathrm{Hom}_{\mathrm{D}(\mathbf{Z})}(\mathbf{Z}/2\mathbf{Z}, (\mathbf{Z}/2\mathbf{Z})[1])$.

PROOF OF THEOREM 2.4. We shall assume the map f has relative dimension equal to n. Consider the cartesian diagram

$$\begin{array}{ccc} \mathrm{X} \times_\mathrm{Y} \mathrm{X} & \xrightarrow{p} & \mathrm{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ \mathrm{X} & \xrightarrow{f} & \mathrm{Y} \end{array}.$$

By proper base change, we have $f^* f_* \mathbf{Q}_\mathrm{X} = p_* \mathbf{Q}_{\mathrm{X}\times_\mathrm{Y}\mathrm{X}}$. By Poincaré duality

$$\mathscr{H}om(\mathrm{R}^k p_* \mathbf{Q}_{\mathrm{X}\times_\mathrm{Y}\mathrm{X}}, \mathbf{Q}_\mathrm{X}) \simeq \mathrm{R}^{2n-k} p_* \mathbf{Q}_\mathrm{X},$$

which in the derived category gives $\mathrm{R}\mathscr{H}om(p_* \mathbf{Q}_{\mathrm{X}\times_\mathrm{Y}\mathrm{X}}, \mathbf{Q}_\mathrm{X}) \simeq p_* \mathbf{Q}_\mathrm{X}[2n]$. It follows that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}^b(\mathrm{Y})}(f_* \mathbf{Q}_\mathrm{X}, f_* \mathbf{Q}_\mathrm{X}) &\simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{X}\times_\mathrm{Y}\mathrm{X})}(p_* \mathbf{Q}_{\mathrm{X}\times_\mathrm{Y}\mathrm{X}}, \mathbf{Q}_\mathrm{X}) \\ &\simeq \mathrm{H}^{2n}(\mathrm{X} \times_\mathrm{Y} \mathrm{X}; \mathbf{Q}). \end{aligned}$$

By Theorem 1.3 and Lemma 3.1, applying to $X \times_Y X \to Y$, we have a surjective map

$$\begin{aligned}\varphi\colon \mathrm{Hom}_{D^b(Y)}(f_*\mathbf{Q}_X, f_*\mathbf{Q}_X) &\to H^0(Y; R^{2n} f_*\mathbf{Q}_X)\\ &\simeq \bigoplus_p H^0(Y, R^{2n-p} f_*\mathbf{Q}_X \boxtimes R^p f_*\mathbf{Q}_X)\\ &\simeq \bigoplus_p \mathrm{Hom}_{D^b(Y)}(R^p f_*\mathbf{Q}_X, R^p f_*\mathbf{Q}_X).\end{aligned}$$

Therefore, we can lift the identity endomorphism of $R^p f_*\mathbf{Q}_X$ to an endomorphism δ_p of $f_*\mathbf{Q}$ in $D^b(Y)$. For each p,

$$\mathscr{H}^i(\delta_p)\colon R^i f_*\mathbf{Q}_X \to R^i f_*\mathbf{Q}_X$$

is precisely the i^{th} component of $\varphi(\delta_p)$ in the displayed decomposition. By our construction it is zero unless $i = p$, in which case it is the identity. The criterion 3.2 applies, and we obtain the desired splitting. □

Theorem 2.4 shows in particular the direct image $f_*\mathbf{Q}$ of any smooth projective morphism $f\colon X \to Y$ between complex manifolds splits in the derived category. According to [7, §4.1, (4.1.1.1)], if f is a smooth *proper* morphism between nonsingular *algebraic varieties*, the splitting $f_*\mathbf{Q}_X \simeq \bigoplus R^m f_*\mathbf{Q}_X[-m]$ still holds. The previous proof remains applicable, assuming that the conclusion of Lemma 3.1 is valid for smooth [5] proper morphisms between nonsingular algebraic varieties. Deligne managed to establish this by reducing the problem to the projective case through the use of mixed Hodge theory.

For a smooth proper morphism $f\colon X \to Y$, Deligne also proves stronger properties of the local systems $R^m f_*\mathbf{Q}_X$.

THEOREM 3.4. *Let $f\colon X \to Y$ be a proper morphism of nonsingular algebraic varieties. Then for any m, $R^m f_*\mathbf{Q}_X$ is a semisimple object in the category* Loc(Y) *of $\mathbf{Q}$-local systems on* Y. (See [7, §4.2]).

Deligne's proof of semisimplicity makes an essential use of his mixed Hodge theory. We shall not delve into details for now.

Recall that the category Loc(Y) corresponds to the category of finite-dimensional representations of the fundamental group $\pi_1(Y)$. Generally, $\pi_1(Y)$ is a complex and intricate abstract group, and its representations are typically not anticipated to be semisimple; while Deligne's semisimplicity theorem asserts that representations of $\pi_1(Y)$ originating from smooth proper families over Y are consistently semisimple. This makes Theorem 3.4 surprising and remarkable.

Combining the splitting $Rf_*\mathbf{Q}_X$ for smooth proper algebraic maps, and the semisimplicity theorem, we get the following theorem.

THEOREM 3.5. *Let $f\colon X \to Y$ be a smooth proper morphism between nonsingular algebraic varieties. Then in the derived category $D^b(Y)$, $f_*\mathbf{Q}_X$ splits into the direct sum of its cohomology sheaves:*

$$f_*\mathbf{Q}_X \simeq \bigoplus R^i f_*\mathbf{Q}_X[-i].$$

Moreover, for every i, there exist simple local systems $\mathscr{L}_1^{(i)}, \dots, \mathscr{L}_{m_i}^{(i)}$ on Y, such that $R^i f_*\mathbf{Q}_X \simeq \mathscr{L}_1^{(i)} \oplus \cdots \oplus \mathscr{L}_{m_i}^{(i)}$.

Theorem 3.5 will be referred to as the "smooth decomposition theorem".

4. Decomposition theorem: overview

Let Y be a space. Then Loc(Y), the category of $\mathbf{Q}$-local systems (i.e., locally constant sheaves of finite dimensional $\mathbf{Q}$-vector spaces) is artinian, noetherian, and admits a duality functor

$$\mathscr{L} \mapsto \mathscr{L}^* = \mathscr{H}om(\mathscr{L}, \mathbf{Q}_Y)\colon \mathrm{Loc}(Y)^{\mathrm{op}} \to \mathrm{Loc}(Y),$$

satisfying $\mathscr{L} \simeq \mathscr{L}^{**}$.

If $f\colon X \to Y$ is a smooth proper morphism of smooth algebraic varieties, recall that the decomposition theorem for f we just discussed says that $R^m f_*(\mathbf{Q}_X) \in \mathrm{Loc}(Y)$, and they are semisimple, and $f_*\mathbf{Q}_X$ splits into the direct sums of $R^m f_*\mathbf{Q}_X[-m]$ in the derived category. More, the Poincaré duality of the fibers implies that $f_*\mathbf{Q}$ is self dual, in the sense that

$$R\mathscr{H}om(f_*\mathbf{Q}_X, \mathbf{Q}_Y) \simeq f_*\mathbf{Q}_X[-2n].$$

If $f\colon X \to Y$ is a proper morphism between algebraic varieties, but we do not assume that X or Y is nonsingular, nor do we assume that f is a smooth morphism, then the sheaves $R^m f_* \mathbf{Q}_X$ are no longer local systems, and $f_*\mathbf{Q}_X$ may not split into a direct sum of shifts of its cohomology sheaves. Since we step away from the abelian category Loc(Y), it is not immediately evident how to formulate "semisimplicity" in this context.

Example 4.1. Let S be the affine cone over a nonsingular projective curve $C \subset \mathbf{P}^2$ of genus g. Denote by σ the vertex of the cone. Let $\pi\colon X \to S$ be the blow up along σ. Then X is nonsingular. In fact, X is the total space of the geometric line bundle associated to the invertible sheaf $\mathcal{O}_C(-1)$ on C. We claim that $\pi_*\mathbf{Q}_X$ is *not* isomorphic to $R^0\pi_*\mathbf{Q}_X \oplus R^1\pi_*\mathbf{Q}_X[-1] \oplus R^2\pi_*\mathbf{Q}_X[-2] = \mathbf{Q}_S \oplus \mathbf{Q}_\sigma^{2g}[-1] \oplus \mathbf{Q}_\sigma[-2]$.

Indeed, we have $R\Gamma_c(S;\pi_*\mathbf{Q}_X) = R\Gamma_c(X;\mathbf{Q}_X)$. Hence the dimensions of its cohomology spaces are $0, 0, 1, 2g, 1$. On the other hand, $R\Gamma_c(\mathbf{Q}_S \oplus \mathbf{Q}_\sigma^{2g}[-1] \oplus \mathbf{Q}_\sigma[-2]) = R\Gamma_c(S;\mathbf{Q}_S) \oplus \mathbf{Q}^{2g}[-1] \oplus \mathbf{Q}[-2]$. In particular, the first cohomology of the complex $\mathbf{Q}_S \oplus \mathbf{Q}_\sigma^{2g}[-1] \oplus \mathbf{Q}_\sigma[-2]$ is nonzero.

Beilinson, Bernstein, and Deligne [1] introduced an abelian full subcategory Perv(X) of $D^b(X)$ for any variety X, called the category of *perverse sheaves* on X, which can replace the role of Loc(Y) in the "singular" setting. They proved a number of properties of this category:

(a) The category Perv(X) of perverse sheaves on X is both an artinian and noetherian category. Additionally, it admits an antiequivalence $\mathbf{D}\colon \mathrm{Perv}(X)^{\mathrm{op}} \to \mathrm{Perv}(X)$, known as *Verdier duality*, satisfying *biduality* $\mathbf{D} \circ \mathbf{D} \simeq \mathrm{Id}$.

(b) If Z is an irreducible closed subset of X, and $\mathscr{L}$ is a local system on some nonsingular Zariski open subset of Z, there exists an object $IC_Z(\mathscr{L})$ in Perv(X), called the *intersection complex of the local system* $\mathscr{L}$. When $\mathscr{L}$ is a simple local system, the corresponding $IC_Z(\mathscr{L})$ is a simple perverse sheaf. Furthermore, it is true that all simple perverse sheaves in Perv(X) can be expressed as $IC_Z(\mathscr{L})$ for some simple local system $\mathscr{L}$ defined on a nonsingular open subset of some irreducible closed subset Z of X. We shall simply write IC_X instead of $IC_X(\mathbf{Q})$. If X is smooth of pure dimension n, then $IC_X = \mathbf{Q}_X[n]$.

(c) Let $\mathscr{L}$ be a local system defined on some nonsingular open subset of an irreducible closed subset Z of X. Then the hypercohomology (resp. hypercohomology with compact support) of $IC_Z(\mathscr{L})$ is called the intersection cohomology (resp. *intersection cohomology with compact support*) of $\mathscr{L}$, denoted as $IH^*_{(c)}(X;\mathscr{L}) := H^*_{(c)}(X; IC_Z(\mathscr{L}))$. Then, we have a perfect pairing, known as *Poincaré duality for intersection cohomology*:

$$IH^i_c(X;\mathscr{L}) \otimes IH^{-i}(X;\mathscr{L}^*) \to \mathbf{Q}.$$

In particular, if X is an irreducible variety, then $IH^*(X;\mathbf{Q})$ satisfies the Poincaré duality.

(d) If $f\colon X \to Y$ is a proper algebraic map between irreducible algebraic varieties, then the direct image sheaf $f_* IC_X$ decomposes into a direct sum

$$f_*(IC_X) \simeq \bigoplus {}^{\mathrm{p}}\mathscr{H}^i f_*(IC_X)[-i],$$

in the derived category. Here ${}^{\mathrm{p}}\mathscr{H}^i$ are the so-called *perverse cohomology sheaves*, and are perverse sheaves. This is an analogue of Theorem 2.4.

(e) Furthermore, in the context of (d), the perverse sheaves ${}^{\mathrm{p}}\mathscr{H}^i f_*(IC_X)$ are *semisimple perverse sheaves*, that is, they are direct sums of simple perverse sheaves in Perv(Y). This is the analogy of Deligne's semisimplicity theorem, Theorem 3.4.

The combination of items (d) and (e) mentioned earlier is famously known as the *Decomposition Theorem*. This theorem was established by Beilinson, Bernstein, Deligne, and Gabber and it can be found in their work "Faisceaux pervers", Théorème 6.2.5. [6]

Notes

1. In the opening of Beilinson, Bernstein, and Deligne [1], the authors said: "It had initially been planned for O. Gabber to be a co-author of this article. However, he preferred to abstain from it in order not to be jointly responsible for any errors or inaccuracies it may contain. Nonetheless, he is still responsible for many ideas that we have used, and the reader owes him numerous critiques which, we hope, have helped improve the manuscript."

2. The hard Lefschetz theorem was formulated in Lefschetz's 1924 memoir *L'analysis situs et la géométrie algébrique*, Chap. V, §7. Lefschetz's formulation of the theorem relied on homology rather than cohomology (which had not yet been developed), and it assumed that M is a projective manifold with ω representing the Poincaré dual of a hyperplane section.

However, the proof that Lefschetz gave was flawed. The first correct proof is due to Hodge [11, §§42–44]. Hodge's proof uses his theory of harmonic forms on compact Kähler manifolds, a theory now recognized as *Hodge theory*. It was also Hodge who established the Lefschetz decomposition, an equivalent manifestation of the hard Lefschetz theorem. The modern formulation of the Lefschetz decomposition we gave in Theorem 1.6 is due to Weil [18].

For smooth projective varieties X over an algebraically closed field k, the hard Lefschetz theorem is still valid for ℓ-adic cohomology, where ℓ is a prime different from the characteristic of k. The proof is based on the global invariant cycle theorem (see Note 5 below) for ℓ-adic cohomology, and the semisimplicity of smooth proper direct images in the ℓ-adic world. Both results are consequences of the theory of weights as developed by Deligne [5].

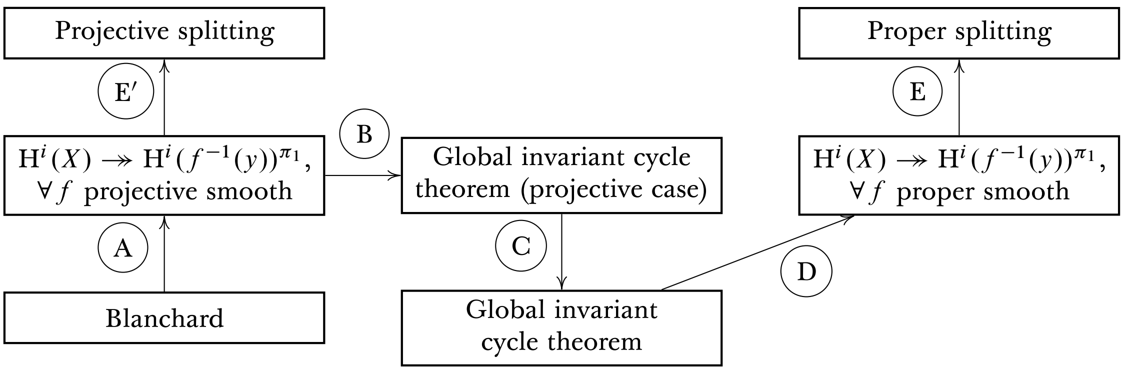
3. Let $\mathcal{A}$ be an abelian category. Then Deligne [6, Proposition 1.2] proved that $\mathrm{K} \in \mathrm{D}^b(\mathcal{A})$ is isomorphic to $\bigoplus \mathrm{H}^i(\mathrm{K})[-i]$ if and only if for any cohomological functor $\mathrm{T}\colon \mathrm{D}^b(\mathcal{A}) \to \mathsf{Ab}$, the "Verdier spectral sequence"

$$\mathrm{E}_2^{p,q} = \mathrm{T}(\mathrm{H}^q\mathrm{K}[p]) \Rightarrow \mathrm{T}(\mathrm{K}[p+q])$$

degenerates at E_2. So splitting in the derived category may be explained as a universal reason for the E_2-degeneration of spectral sequences.

4. Deligne [6] explains, in an abstract framework, how a suitable relative hard Lefschetz theorem leads to a splitting in the derived category. The first complex analytic proof of decomposition theorem by Morihiko Saito [16] used this criterion, by first prove relative hard Lefschetz theorem for Hodge Modules. The method used by Beilinson, Bernstein, and Deligne [1] is different, and famously relies on the theory of weights for algebraic varieties over finite fields. The relative hard Lefschetz is *deduced* from the decomposition theorem in the approach of Beilinson, Bernstein, and Deligne.

5. The chain of logical implications concerning the splitting for a smooth proper morphism between smooth varieties can be represented by the following diagram.



We have explained the implications A and E (= E′) in the main text. The *global invariant cycle theorem* (also known as the *theorem of the fixed part*) asserts that if $f\colon \mathrm{X} \to \mathrm{Y}$ is a proper smooth morphism of nonsingular algebraic varieties, then for *any* proper smooth compactification $\overline{\mathrm{X}}$ of X, the restriction map $\mathrm{H}^i(\overline{\mathrm{X}}) \to \mathrm{H}^i(\mathrm{X}) \to \mathrm{H}^i(f^{-1}(y))^{\pi_1}$ is surjective. The implication B crucially relies on mixed Hodge theory, see [7, Corollarie (3.2.18)]. The implication C uses resolution of singularities, Poincaré duality and Chow's lemma, and D is trivial. Going through a smooth compactification is essential for the argument $\mathrm{D} \circ \mathrm{C} \circ \mathrm{B}$.

While for a projective (or even Kähler) morphism, the splitting theorem is true, Deligne's proper splitting theorem cannot be generalized to proper smooth maps between complex manifolds, as shown by the Hopf surface example (Example 1.2).

6. There is a more general version of the decomposition theorem than stated in §4. It states that if $f\colon \mathrm{X} \to \mathrm{Y}$ is a proper morphism of varieties over $\mathbf{C}$, and $\mathcal{F}$ is any semisimple perverse sheaf on X (thus $\mathcal{F}$ is a direct sum of intersection complexes of simple local systems on some nonsingular locally closed subset of X), then $f_*\mathcal{F}$ can be written as a direct sum $f_*\mathcal{F} \simeq \bigoplus_i IC_Y(\mathscr{L}_i)[d_i]$, where $\mathscr{L}_i$ are simple local systems defined on some nonsingular locally closed subset U_i of Y. This was conjectured by Kashiwara [12], and proved by Böckle and Khare [3] (thanks to Drinfeld [8]) using ℓ-adic cohomology, and by Sabbah [15] and Mochizuki [14] using complex analytic methods.

References

[1] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, in: *Analysis and topology on singular spaces, I (Luminy, 1981)*, vol. 100, Astérisque, Soc. Math. France, Paris, 1982, pp. 5–171 (cit. on pp. 1, 4, 8–10).

[2] André Blanchard, *Sur les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. (3), pp. 157–202, 1956 (cit. on p. 2).

[3] Gebhard Böckle and Chandrashekhar Khare, *Mod l representations of arithmetic fundamental groups. II. A conjecture of A. J. de Jong*, Compos. Math. Vol. 142, no. 2, pp. 271–294, 2006 (cit. on p. 10).

[4] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, vol. 82, Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982, pp. xiv+331 (cit. on p. 4).

[5] Pierre Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. No. 52, pp. 137–252, 1980 (cit. on p. 9).

[6] Pierre Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. No. 35, pp. 259–278, 1968 (cit. on pp. 3, 4, 6, 9).

[7] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. No. 40, pp. 5–57, 1971 (cit. on pp. 7, 9).

[8] Vladimir Drinfeld, *On a conjecture of Kashiwara*, Math. Res. Lett. Vol. 8, no. 5-6, pp. 713–728, 2001 (cit. on p. 10).

[9] Phillip A. Griffiths and Joseph Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978, pp. xii+813 (cit. on p. 5).

[10] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002, pp. xii+544 (cit. on p. 4).

[11] W. V. D. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, Cambridge; The Macmillan Company, New York, 1941, pp. ix+281 (cit. on p. 9).

[12] Masaki Kashiwara, *Semisimple holonomic D-modules*, in: *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, vol. 160, Progr. Math. Birkhäuser Boston, Boston, MA, 1998, pp. 267–271 (cit. on p. 10).

[13] S. Lefschetz, *L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1950, pp. vi+154 (cit. on p. 9).

[14] Takuro Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor* D*-modules. I*, Mem. Amer. Math. Soc. Vol. 185, no. 869, pp. xii+324, 2007 (cit. on p. 10).

[15] Claude Sabbah, *Polarizable twistor D-modules*, Astérisque, no. 300, pp. vi+208, 2005 (cit. on p. 10).

[16] Morihiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. Vol. 24, no. 6, 849–995 (1989), 1988 (cit. on p. 9).

[17] Claire Voisin, *Théorie de Hodge et géométrie algébrique complexe*, vol. 10, Cours Spécialisés [Specialized Courses], Société Mathématique de France, Paris, 2002, pp. viii+595 (cit. on pp. 2, 4, 5).

[18] André Weil, *Introduction à l'étude des variétés kählériennes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267, Hermann, Paris, 1958, p. 175 (cit. on pp. 2, 9).

> Perverse sheaves are neither sheaves nor perverse in the usual sense, so the terminology requires an explanation. The word "perverse" may not resonate well with some of us. It originates from the term "perversity". In defining the intersection homology ${}^{\mathrm{p}}\mathrm{IH}_*(\mathrm{X})$, the "perversity" p indicates how much we allow cycles to deviate from transversality to a stratification.
>
> Why "sheaf"? Because if we work with complex coefficients, and X is a purely n-dimensional smooth variety, the analytic de Rham complex functor $\mathcal{M} \mapsto \Omega^*(\mathcal{M})[n]$ is an equivalence between the category of algebraic holonomic modules with regular singularities (including at infinity) on X and the category of perverse sheaves on X.
>
> — [1, Introduction]

Email address: zhangdingxin13@gmail.com