Artin–Grothendieck vanishing theorem

Contents

In this note we prove the Artin–Grothendieck vanishing theorem for constructible sheaves. Although the theorem works well in more abstract contexts, we shall treat exclusively *complex* algebraic varieties with classical topology.

1. Recollections

The first group of players in our arena is local systems. By a *local system* on a variety X, I mean a sheaf of finite dimensional rational vector spaces which is locally constant. More precisely, a local system is a sheaf $U \mapsto L(U)$ with the following properties:

(i) for each open U, L(U) is a vector space over the field of rational numbers,
(ii) every point of X has a neighborhood U, such that for any $U' \subset V$, $L(U') = E$, where E is some finite dimensional rational vector space, and if $U'' \subset U'$, the restriction map $E = L(U') \to L(U'') = E$ is the identity map.

Of course, I can replace "rational" by "complex", so I get the notion of local systems of *complex* vector spaces, and you should have no difficulty to imagine their other cousins. Constant sheaves of vector spaces are examples of local systems. I will provide more examples of local systems a bit later.

I want to make an immediate warning that unless X is a discrete set of points, there is no nonzero analytic coherent $\mathcal{O}_X$-Module $\mathcal{F}$ that is simultaneously also a local system (of complex vector spaces). On the other hand, if L is a local system, then $L \otimes_{\mathbf{C}} \mathcal{O}_X$ is an analytic locally free coherent $\mathcal{O}_X$-Module.

There are primarily two methods for generating local systems on algebraic varieties (see *Local systems and integrable connections*, Remark 1.2). The first method involves linear differential equations with polynomial coefficients. Consider, for instance, the second-order ordinary differential equation:

$$\left[\left(t\frac{d}{dt}\right)^2 - t^4\left(t\frac{d}{dt}+1\right)\left(t\frac{d}{dt}+3\right)\right]F(t) = 0. \tag{1.1}$$

This equation is defined on the affine line and is non-singular away from $t = 0,1$. Let L(U) denote the set of solutions of this ordinary differential equation for any $U \subset \mathbf{C} - \{0,1\}$. According to the existence and uniqueness theorem of ordinary differential equations, for any simply connected open set $\Delta \subset \mathbf{C} - \{0,1\}$, the vector space $L(\Delta)$ is two-dimensional, and the assignment $U \mapsto L(U)$ forms a local system.

Another approach is through the Ehresmann lemma, as discussed in, for example, Voisin 2002, Theorem 9.3. This lemma states that if $f\colon X \to Y$ is a proper and smooth map (i.e., f is a submersion in the C^∞ sense) between nonsingular manifolds, then f forms a locally topologically trivial fibration. According to the generalized Vietoris–Begle theorem (see *Proper base change in topology*, Proposition 4.4), $R^i f_* \mathbf{Q}_X$ is a local system on Y.

For those familiar with algebraic Gauss–Manin connections, it's worth noting that if X and Y are algebraic varieties, $\mathrm{R}^i f_* \mathbf{Q}$ can also be interpreted as the sheaf of solutions to certain algebraic linear partial differential equations.

A more flexible type of sheaves than local systems is constructible sheaves. We say a sheaf $\mathcal{F}$ of $\mathbf{Q}$-vector spaces on an algebraic variety X is *Zariski-constructible* (or simply *constructible* when without ambiguity) if there is a sequence of Zariski-closed subsets $\emptyset = \mathrm{X}_0 \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_r = \mathrm{X}$, such that $\mathcal{F}|_{\mathrm{X}_i - \mathrm{X}_{i-1}}$ is a local system (on the quasi-projective variety $\mathrm{X}_i - \mathrm{X}_{i-1}$). We shall use $\mathrm{D}^b_c(\mathrm{X})$ to denote the full subcategory of the $\mathrm{D}^+(\mathrm{X})$ consisting of complexes satisfying the following two conditions:

- ◇ $\mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for all $|i| \gg 0$, and
- ◇ $\mathcal{H}^i(\mathcal{F}^\bullet)$ are constructible sheaves.

I sometimes will call objects of $\mathrm{D}^b_c(\mathrm{X})$ "constructible complexes". Also, I will drop the bullets in the superscripts for the ease of typing.

The constructible derived category $\mathrm{D}^b_c(\mathrm{X})$ is a convenient framework for exploiting topological properties of *algebraic* varieties. (Constructible sheaf theory on complex analytic spaces requires more careful consideration.)

The category $\mathrm{D}^b_c(\mathrm{X})$ has a number of permanent properties, making it a safe walled garden for most of our games. Here are some:

(i) Let X be a variety. If $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is a distinguished triangle, and two of the three terms are in $\mathrm{D}^b_c(\mathrm{X})$, then so is the third (Exercise).

(ii) Let $f \colon \mathrm{X} \to \mathrm{Y}$ be a morphism of algebraic varieties. Then $f^*(\mathrm{D}^b_c(\mathrm{Y})) \subset \mathrm{D}^b_c(\mathrm{X})$, $\mathrm{R}f_*(\mathrm{D}^b_c(\mathrm{X})) \subset \mathrm{D}^b_c(\mathrm{Y})$ and $f_!(\mathrm{D}^b_c(\mathrm{X})) \subset \mathrm{D}^b_c(\mathrm{Y})$. This is nontrivial. See Achar 2021, Theorem 2.7.1.

(iii) If $i \colon \mathrm{Z} \to \mathrm{X}$ is a closed immersion of algebraic varieties. Then $i^! \mathrm{D}^b_c(\mathrm{X}) \subset \mathrm{D}^b_c(\mathrm{Z})$ (Exercise).

(iv) If $\mathcal{F}$ and $\mathcal{G}$ are constructible complexes, then $\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ are in $\mathrm{D}^b_c(\mathrm{X})$ (Exercise).

Further properties will emerge as we delve into discussions on Verdier duality.

2. Beilinson's "basic lemma" and applications

The following theorem is a special case of Beilinson 1987, Lemma 3.3. We will defer its proof to next section. In this section, we discuss some of its immediate consequences.

THEOREM 2.1. *Let* X *be an affine variety of dimension* n. *Let* $\mathcal{F}$ *be a constructible sheaf on* X. *Then there is a non-empty open subset* U *of* X, *such that* $\mathrm{Z} = \mathrm{X} - \mathrm{U}$ *has dimension* $\leq n-1$, *and*

$$\mathrm{H}^i(\mathrm{X}, \mathrm{Z}; \mathcal{F}) = 0$$

for any $i \neq n$. *In particular, the restriction* $\mathrm{H}^i(\mathrm{X}) \to \mathrm{H}^i(\mathrm{Z})$ *is an isomorphism for* $i < n-1$, *and an injective map for* $i = n-1$.

REMARK 2.2. As we will see from the proof below, Z may be taken to be a generic hyperplane section with respect to any closed embedding $\mathrm{X} \to \mathbf{A}^{\mathrm{N}}$.

REMARK 2.3 (Cellular cohomology of constructible sheaves). Let X be an affine variety of dimension n. Let $\mathcal{F}$ be a constructible sheaf on X. Then the basic lemma, Theorem 2.1, and Remark 2.2 imply that there is a sequence of closed subvarieties of X:

$$\mathrm{X} = \mathrm{X}_n \supset \mathrm{X}_{n-1} \supset \cdots \supset \mathrm{X}_0 \supset \mathrm{X}_{-1} = \emptyset$$

such that $\dim \mathrm{X}_i = i$, and

$$\mathrm{H}^m(\mathrm{X}_i, \mathrm{X}_{i-1}; \mathcal{F}|_{\mathrm{X}_i}) = 0$$

for all $m \neq i$. Let $\mathrm{C}^i(\mathcal{F}) = \mathrm{H}^i(\mathrm{X}_i, \mathrm{X}_{i-1}; \mathcal{F}|_{\mathrm{X}_i})$. Define $d \colon \mathrm{C}^i(\mathcal{F}) \to \mathrm{C}^{i+1}(\mathcal{F})$ via the following diagram

$$\begin{array}{ccccc} \mathrm{C}^i(\mathcal{F}) & & \xrightarrow{\quad d \quad} & & \mathrm{C}^{i+1}(\mathcal{F}) \\ \| & & & & \| \\ \mathrm{H}^i(\mathrm{X}_i, \mathrm{X}_{i-1}; \mathcal{F}|_{\mathrm{X}_i}) & \xrightarrow{\mathrm{res}} & \mathrm{H}^i(\mathrm{X}_i; \mathcal{F}|_{\mathrm{X}_i}) & \xrightarrow{\delta} & \mathrm{H}^{i+1}(\mathrm{X}_{i+1}, \mathrm{X}_i; \mathcal{F}|_{\mathrm{X}_{i+1}}) \end{array}.$$

By a spectral sequence argument (exercise), one checks that $d^2 = 0$, and that $\mathrm{H}^m(\mathrm{C}^\bullet(\mathcal{F})) = \mathrm{H}^m(\mathrm{X};\mathcal{F})$. This is an analogue, in algebraic geometry, of the fact that the cellular cohomology of a CW complex is isomorphic to its singular cohomology.

THEOREM 2.4 (Artin–Grothendieck (SGA 4, Exposé XIV, Corollaire 3.5)). *Let* X *be an affine variety of dimension* n*. Let* $\mathcal{F}$ *be a constructible sheaf on* X*. Then* $\mathrm{H}^i(\mathrm{X};\mathcal{F}) = 0$ *for any* $i > n$*.* [1]

PROOF. This follows easily from Remark 2.3. Alternatively it is also straightforward to prove directly. We use the basic lemma (Theorem 2.1) coupled with a noetherian induction. Let U be as in the basic lemma. Then we have an exact sequence of cohomology

$$\cdots \to \mathrm{H}^i(\mathrm{X};\mathcal{F}) \to \mathrm{H}^i(\mathrm{Z};\mathcal{F}|_{\mathrm{Z}}) \xrightarrow{\delta^i} \mathrm{H}^{i+1}(\mathrm{X},\mathrm{Z};\mathcal{F}) \to \cdots .$$

The basic lemma says that $\mathrm{H}^i(\mathrm{X},\mathrm{Z};\mathcal{F}) = 0$ for all $i \neq n$. The inductive hypothesis says that $\mathrm{H}^i(\mathrm{Z};\mathcal{F}|_{\mathrm{Z}}) = 0$ for all $i < n$. Therefore, for $i > n$, we have $\mathrm{H}^i(\mathrm{X};\mathcal{F}) \cong \mathrm{H}^i(\mathrm{Z};\mathcal{F}) = 0$. □

COROLLARY 2.5. *Let* X *be a nonsinglular projective variety of pure dimension* n*. Let* $\mathrm{Y} \subset \mathrm{X}$ *be an ample divisor. Then*

$$\mathrm{H}^i(\mathrm{X};\mathbf{Q}) \to \mathrm{H}^i(\mathrm{Y};\mathbf{Q})$$

is injective for $i = n-1$*, and bijective for* $i < n-1$*.*

PROOF. It suffices to prove $\mathrm{H}^i_c(\mathrm{X}-\mathrm{Y};\mathbf{Q}) = 0$ for $i < n$. Since X is nonsingular, so is $\mathrm{X}-\mathrm{Y}$. By Poincaré duality, it suffices to show $\mathrm{H}^j(\mathrm{X}-\mathrm{Y};\mathbf{Q}) = 0$ for $j > n$. Since Y is ample, $\mathrm{X}-\mathrm{Y}$ is affine. We can now apply Theorem 2.4. □

Artin's vanishing theorem (Theorem 2.4) actually implies a vanishing theorem for a wider range of objects in $\mathrm{D}^b_c(\mathrm{X})$. These objects are closely related to perverse sheaves.

DEFINITION 2.6. A constructible complex $\mathcal{F} \in \mathrm{D}^b_c(\mathrm{X})$ is said to satisfy the *support condition*, if

$$\boxed{\dim\{x \in \mathrm{X} : \mathcal{H}^i(\mathcal{F})_x \neq 0\} \leq -i}.$$

(In short: $\dim \operatorname{Supp} \mathcal{H}^i(\mathcal{F}) \leq -i$.) Recall: $\mathcal{H}^i(\mathcal{F})_x = \operatorname{colim}_{\mathrm{U}} \mathrm{H}^i(\mathrm{U},\mathcal{F}|_{\mathrm{U}})$, where the colimit is taken over all the neighborhoods of x. Denote by ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}_{\mathrm{X}}$ the full subcategory of $\mathrm{D}^b_c(\mathrm{X})$ consisting of objects satisfying the support condition.

In certain literature, a constructible complex satisfying the support condition is often referred to as a semi-perverse sheaf. This terminology highlights the partial fulfillment of conditions typically associated with perverse sheaves. A *perverse sheaf* goes a step further: it is a semi-perverse sheaf whose Verdier dual (see *Verdier duality*, Remark 3.13) is also semi-perverse.

EXAMPLE 2.7. Assume that $\dim \mathrm{X} \leq n$, and $\mathcal{F}$ is a constructible sheaf. Then $\mathcal{F}[n]$ satisfies the support condition.

PROPOSITION 2.8 (Artin vanishing for semi-perverse sheaves). *Let* X *be an affine algebraic variety of dimension* n*. Let* $\mathcal{F}$ *be an object of* ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}_{\mathrm{X}}$*, i.e.,* $\mathcal{F}$ *satisfies the support condition. Then for any* $i > 0$*, we have* $\mathrm{H}^i(\mathrm{X};\mathcal{F}) = 0$.

PROOF. Since $\mathcal{H}^q(\mathcal{F})$ are constructible sheaves, and since X is affine, it follows from the support condition and Theorem 2.4 that

$$\mathrm{H}^p(\mathrm{X};\mathcal{H}^q(\mathcal{F})) = 0 \text{ if } p > -q. \tag{2.9}$$

But there is a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{X};\mathcal{H}^q(\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{X};\mathcal{F}),$$

so (2.9) implies that $\mathrm{H}^i(\mathrm{X};\mathcal{F}) = 0$ for any $i > 0$. □

3. Proof of Theorem 2.1

I will now present Nori's proof of the basic lemma. As previously mentioned, the subtlety of working with sheaves lies in being attentive to whether certain base change maps are isomorphisms. I aim to demonstrate this subtlety through the proof of the basic lemma.

1. Reduction to $\mathbf{A}^n$. By Noether normalization, there is a finite morphism $\pi\colon X \to \mathbf{A}^n$ (in effect, π can be taken as a generic linear projection $\mathbf{A}^N \to \mathbf{A}^n$, if $X \subset \mathbf{A}^N$ is Zariski closed). Since π is finite, $\pi_*\mathcal{F}$ remains a constructible sheaf on $\mathbf{A}^n$, and $R^i\pi_*\mathcal{F} = 0$ for $i \neq 0$. Suppose we have shown the basic lemma for $\mathbf{A}^n$. Then there is a nonempty open $a\colon V \to \mathbf{A}^n$ of $\mathbf{A}^n$ such that

$$H^i(\mathbf{A}^n; a_!a^*\pi_*\mathcal{F}) = H^i(\mathbf{A}^n, \mathbf{A}^n - V; \pi_*\mathcal{F}) = 0 \quad \text{for any } i \neq n.$$

Form the cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ \downarrow{\scriptstyle \varpi} & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{a} & \mathbf{A}^n \end{array}.$$

Then ϖ is finite as well. The finiteness implies that $\varpi_* = \varpi_!$ and $\pi_* = \pi_!$. Ergo, base change is allowed:

$$a_!a^*\pi_* = a_!\varpi_*j^* = \pi_*j_!j^*.$$

Therefore

$$\begin{aligned} H^i(X, Z; \mathcal{F}) &= H^i(X, j_!j^*\mathcal{F}) \\ &= H^i(\mathbf{A}^n, \pi_*j_!j^*\mathcal{F}) \\ &= H^i(\mathbf{A}^n, a_!a^*\pi_*\mathcal{F}) \\ &= H^i(\mathbf{A}^n, \mathbf{A}^n - V; \pi_*\mathcal{F}) \\ &= 0 \quad (\text{for } i \neq n). \end{aligned}$$

2. Fibration. Now we can assume $X = \mathbf{A}^n$. Let f be a polynomial such that the restriction of $\mathcal{F}$ to the distinguished open $D(f)$ is a local system. By changing coordinates, the projection $(x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)\colon \mathbf{A}^n \to \mathbf{A}^{n-1}$ is a finite morphism on the hypersurface $\{f = 0\}$. Let $b\colon D(f) \to \mathbf{A}^n$ be the inclusion map. Let $\mathcal{F}_1 = b_!b^*\mathcal{F}$. Then $\mathcal{F}$ fits into an exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is supported on the hypersurface $V(f)$ cutout by f. Since $\mathcal{F}_1$ is extended by zero along $V(f)$, $\mathcal{F}|_{V(f)} = 0$. I claim that it suffices to find an open subset $U \subset D(f)$ such that

$$H^n(\mathbf{A}^n, \mathbf{A}^n - U; \mathcal{F}_1) = 0.$$

Indeed, denote by j the inclusion $U \to D(f) \to \mathbf{A}^n$, then we have $j_!\mathcal{F}|_U$ and $j_!\mathcal{F}_1|_U$. Therefore, without loss of generality, I may assume that $\mathcal{F}|_{V(f)} = 0$, and $\mathcal{F}|_{D(f)}$ is a local system.

3. Base change. My next step involves pushing $\mathcal{F}$ down to $\mathbf{A}^{n-1}$ and applying the inductive hypothesis. However, to establish the connection between the objects upstairs and downstairs, I must address the base change problem. To facilitate the exposition, I will first state a lemma. Once the main result is established using this lemma, I will revisit and examine its validity.

LEMMA 3.1. *Let* B *be an algebraic variety. Let* $g\colon \mathbf{A}^1 \times B \to B$ *be the projection map sending* (x, b) *to* b. *Let* V *be a hypersurface in* $\mathbf{A}^1 \times B$ *such that* $g|_V\colon V \to B$ *is a finite surjective morphism. Let* $\mathcal{F}$ *be a constructible sheaf on* $\mathbf{A}^n$ *such that*

- ◇ $\mathcal{F}|_{D(f)}$ *is a local system, and*
- ◇ $\mathcal{F}|_{V(f)} = 0$.

Then $R^ig_*\mathcal{F} = 0$ *for all* $i \neq 1$, *and for any* y *in* B,

$$(R^1g_*\mathcal{F})_y = H^1(g^{-1}(y); \mathcal{F}|_{g^{-1}(y)}).$$

My arrangements for $g\colon \mathbf{A}^n \to \mathbf{A}^{n-1}$ and $\mathcal{F}$ allow me to apply Lemma 3.1 to $B = \mathbf{A}^{n-1}$. For the constructible sheaf $\mathcal{G} = R^1g_*\mathcal{F}$ on $\mathbf{A}^{n-1}$, inductive hypothesis tells me that I can find a nonempty open set

$j'\colon \mathrm{U}' \to \mathbf{A}^{n-1}$, such that $\mathrm{H}^m(\mathbf{A}^{n-1}; j'_! j'^* \mathcal{G}) = 0$ for all $m \neq n-1$. Let $\mathrm{U} = g^{-1}(\mathrm{U}') \cap \mathrm{D}(f)$, and $j\colon \mathrm{U} \to \mathbf{A}^n$ be the inclusion map. Let us take a look of the following diagram

$$\begin{array}{ccccccc} \mathrm{U} & \longrightarrow & g^{-1}(\mathrm{U}') & \xrightarrow{\beta} & \mathbf{A}^n & \xleftarrow{\alpha} & g^{-1}(\mathrm{Z}) \\ & & \downarrow & & \downarrow g & & \downarrow h \\ & & \mathrm{U}' & \xrightarrow{j'} & \mathbf{A}^{n-1} & \xleftarrow{i'} & \mathrm{Z}' \end{array} \qquad (j\colon \mathrm{U} \to \mathbf{A}^n),$$

where $\mathrm{Z}' = \mathbf{A}^{n-1} - \mathrm{U}'$. Note that the right two squares are cartesian.

Remember that $\mathcal{F}$ is the sheaf obtained by extending a local system on $\mathrm{D}(f)$ by zero. So $\beta^*\mathcal{F}$ is supported entirely on U, and has zero stalk along $\mathrm{V}(f) \cap g^{-1}(\mathrm{U})$. So $j_! j^* \mathcal{F}$ is the same as $\beta_! \beta^* \mathcal{F}$. In order to prove $\mathrm{H}^m(\mathbf{A}^n; j_! j^* \mathcal{F}) = 0$ for $m \neq n$, I can prove instead $\mathrm{H}^m(\mathbf{A}^n; \beta_! \beta^* \mathcal{F}) = 0$, or, equivalently $\mathrm{H}^m(\mathbf{A}^{n-1}; \mathrm{R}g_*(\beta_! \beta^* \mathcal{F})) = 0$, for any $m \neq n$.

On $\mathbf{A}^n$ we have an distinguished triangle

$$\beta_! \beta^* \mathcal{F} \to \mathcal{F} \to \alpha_* \alpha^* \mathcal{F}.$$

Pushing it down to $\mathbf{A}^{n-1}$ gives a distinguished triangle

$$\mathrm{R}g_* \beta_! \beta^* \mathcal{F} \to \mathrm{R}g_* \mathcal{F} \to \mathrm{R}g_* \alpha_* \alpha^* \mathcal{F} = i'_* \mathrm{R}h_* \alpha^* \mathcal{F}.$$

On $\mathbf{A}^{n-1}$ there is another triangle

$$j'_! j'^* \mathrm{R}g_* \mathcal{F} \to \mathrm{R}g_* \mathcal{F} \to i'_* i'^* \mathrm{R}g_* \mathcal{F}.$$

The two triangles on $\mathbf{A}^{n-1}$ are connected by the base change maps:

$$\begin{array}{ccccc} \mathrm{R}g_* \beta_! \beta^* \mathcal{F} & \longrightarrow & \mathrm{R}g_* \mathcal{F} & \longrightarrow & i'_* \mathrm{R}h_* \alpha^* \mathcal{F} \\ \downarrow & & \| & & \downarrow \\ j'_! j'^* \mathrm{R}g_* \mathcal{F} & \longrightarrow & \mathrm{R}g_* \mathcal{F} & \longrightarrow & i'_* i'^* \mathrm{R}g_* \mathcal{F} \end{array}$$

If we can show the left vertical arrow is an isomorphism, we will win. To show the left one is an isomorphism, it suffices to prove the right vertical one is an isomorphism. But then I can apply Lemma 3.1 again, this time with $(g, \mathrm{B}) = (h, \mathrm{Z}')$. Therefore,

$$\begin{aligned} \mathrm{H}^m(\mathbf{A}^n; \beta_! \beta^* \mathcal{F}) &= \mathrm{H}^m(\mathbf{A}^{n-1}; \mathrm{R}g_* \beta_! \beta^* \mathcal{F}) \\ &= \mathrm{H}^m(\mathbf{A}^{n-1}; j'_! j'^* \mathrm{R}g_* \mathcal{F}) \\ &= \mathrm{H}^{m-1}(\mathbf{A}^{n-1}; j'_! j'^* \mathrm{R}^1 g_* \mathcal{F}) \\ &= 0 \end{aligned}$$

for all $m \neq n$.

Proof of Lemma 3.1. The lemma actually works fairly generally for reasonable topological spaces. Since the problem is local, I can assume that B is a compact (necessarily Hausdorff) contractible space. Then for each b in B there is a radius R such that the points of $\mathrm{V} \cap \{(z, b) : |z| \geq \mathrm{R}(b)\}$ is empty. By the compactness of B, I can choose a uniform R that works for any b. Then I can decompose $\mathbf{A}^1 \times \mathrm{B}$ as a union $\{\mathrm{A} \cup \mathrm{E}$, where $\mathrm{A} = \{(z, b) : |z| \leq \mathrm{R}\}$. and $\mathrm{E} = \{(z, b) : |z| \geq \mathrm{R}\}$. The intersection $\mathrm{A} \cap \mathrm{E}$ is homeomorphic to $\mathbf{S}^1 \times \mathrm{B}$ as a space over B. The validity of the base change follows from the closed Mayer–Vietoris theorem, the isomorphy for that of $\mathcal{F}|_{\mathrm{A}}$ (proper base change), $\mathcal{F}|_{\mathrm{A} \cap \mathrm{E}}$ (proper base change), and $\mathcal{F}|_{\mathrm{E}}$ ($\mathrm{E} \to \mathrm{B}$ is a fiber bundle, and $\mathcal{F}$ is a local system on E).

Finally I have to show the vanishing of R^0 and $\mathrm{R}^{i>1}$. For R^0 this is straightforward. For $\mathrm{R}^{>1}$, from the base change isomorphism I only need to show the following lemma.

Lemma 3.2. *Let* U *be a Zariski open subset of* $\mathbf{A}^1$. *Let* $\mathcal{E}$ *be a locally constant sheaf on* U. *Assume that* $\mathrm{Card}(\mathbf{A}^1 - \mathrm{U}) \geq 1$. *Then* $\mathrm{H}^m(\mathbf{A}^1; j_! \mathcal{E}) = 0$ *for* $m \neq 1$.

Proof of Lemma 3.2. We use induction on the cardinality of $\mathbf{A}^1 - \mathrm{U}$. If $\mathrm{Card}(\mathbf{A}^1 - \mathrm{U}) = 1$, then without loss of generality we can assume that $\mathrm{U} = \mathbf{A}^1 - \{0\}$. Since $\pi_1(\mathrm{U})$ is abelian, $\mathcal{E}$ is an iterated extension of rank one local systems. Therefore it suffices to prove the assertion for rank one local systems. If $\mathcal{E} = \mathbf{Q}$, the assertion is obvious. If $\mathcal{E}$ is of rank one and non-constant, then $\mathrm{H}^2(\mathbf{A}^1; j_!\mathcal{E}) = \mathrm{H}^2(\mathbf{A}^1; \mathrm{R}j_*\mathcal{E}) = \mathrm{H}^2(\mathbf{A}^1 - \{0\}; \mathcal{E}) = 0$.

Now assume $\mathrm{Card}(\mathbf{A}^1 - \mathrm{U}) = k+1 > 1$. Let $s \notin \mathrm{U}$ be the point not U with the largest radius. Without loss of generality I shall assume such an s is unique and $|s| = \mathrm{R}$. Let me set up some notation.

- ⋄ Let $\mathrm{D} = \{|z| \le \mathrm{R}\}$. Let $i\colon \mathrm{D} \to \mathbf{A}^1$ be the inclusion map.
- ⋄ Let $\mathrm{E} = \{|z| > \mathrm{R}\}$. Let $h\colon \mathrm{E} \to \mathbf{A}^1$ be the inclusion map.
- ⋄ Let $\mathrm{D}^\circ = \{|z| < \mathrm{R}\}$. Let $j^\circ\colon \mathrm{D}^\circ \to \mathrm{D}$ be the inclusion map.
- ⋄ Let $\mathbf{S} = \{|z| = \mathrm{R}\}$. Let $\iota\colon \mathbf{S} \to \mathrm{D}$ be the inclusion map.

Consider the following diagram in which the row and the column are distinguished triangles.

$$
\begin{array}{ccccc}
 & & & & i_*\iota_*\iota^!(i^*j_!\mathcal{E}) \\
 & & & & \downarrow \\
h_!h^*j_!\mathcal{E} & \longrightarrow & j_!\mathcal{E} & \longrightarrow & i_*i^*(j_!\mathcal{E}) \\
 & & & & \downarrow \\
 & & & & i_*\mathrm{R}j^\circ_*j^{\circ *}(i^*j_!\mathcal{E})
\end{array}
$$

By the five lemma, to prove $\mathrm{H}^m(\mathbf{A}^1; j_!\mathcal{E}) = 0$ for $m \neq 1$, it suffices to prove

(a) $\mathrm{H}^m(\mathbf{A}^1; h_!h^*j_!\mathcal{E}) = 0$ for $m \neq 1$, and
(b) $\mathrm{H}^m(\mathbf{A}^1; i_*\iota_*\iota^!(i^*j_!\mathcal{E})) = 0$ for $m \neq 1$, and
(c) $\mathrm{H}^m(\mathbf{A}^1; i_*\mathrm{R}j^\circ_*j^{\circ *}(i^*j_!\mathcal{E})) = 0$ for $m \neq 1$.

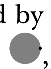
Let me first prove (a). I have a proper differentiable map from $q\colon \mathbf{A}^1 \to \mathbf{A}^1$ which collapses the disk D to a point, giving rise to the following diagram

$$
\begin{array}{ccccc}
\mathrm{D} & \longrightarrow & \mathbf{A}^1 & \xleftarrow{\ h\ } & \mathrm{E} \\
\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q|_{\mathrm{E}}} \\
\{0\} & \longrightarrow & \mathbf{A}^1 & \xleftarrow{\ \alpha\ } & \mathbf{G}_{\mathrm{m}}.
\end{array}
$$

Apparently q is proper and $h|_{\mathrm{E}}$ is a diffeomorphism. Hence $q_* = q_!$, $h|_{\mathrm{E}!} = h|_{\mathrm{E}*}$, and $h|_{\mathrm{E}!}$ is an equivalence between $\mathrm{D}^+(\mathrm{E})$ and $\mathrm{D}^+(\mathbf{G}_{\mathrm{m}})$. We have

$$
\begin{aligned}
\mathrm{H}^m(\mathbf{A}^1; h_!h^*j_!\mathcal{E}) &= \mathrm{H}^m(\mathbf{A}^1; q_*h_!h^*j_!\mathcal{E}) \\
&= \mathrm{H}^m(\mathbf{A}^1; \alpha_!q|_{\mathrm{E}!}h^*j_!\mathcal{E}).
\end{aligned}
$$

Since $\mathrm{E} \cap \mathrm{S} = \emptyset$, $h^*j_!\mathcal{E}$ is a local system on E. Since $q|_{\mathrm{E}}$ is a diffeomorphism, $\mathcal{F} = q|_{\mathrm{E}!}h^*j_!\mathcal{E}$ remains a local system on $\mathbf{G}_{\mathrm{m}}$. By the argument in the paragraph at the beginning of the proof, we have $\mathrm{H}^m(\mathbf{A}^1; \alpha_!\mathcal{F}) = 0$ for $m \neq 1$. This proves (a).

Proof of (b). We have $\mathrm{H}^m(\mathbf{A}^1; i_*\iota_*\iota^!(i^*j_!\mathcal{E})) = \mathrm{H}^m(\mathbf{S}; \iota^!(i^*j_!\mathcal{E}))$. I claim that $\iota^!(i^*j_!\mathcal{E}) = \mathbf{Q}_{\{s\}}^{\mathrm{rank}(\mathcal{E})}[-1]$ is a skyscraper sheaf, shifted to right, supported at the point s. Once this is established, (b) will follow immediately. To prove the claim, write for convenience $\mathcal{G} = i^*j_!\mathcal{E}$. Then in a small neighborhood of $\mathbf{S}$, $\mathcal{G}$ is a local system, except at $\{s\}$, at which it is extended by zero. For each $x \in \mathbf{S}$, $x \neq s$, we can choose sufficiently small neighborhood $\mathrm{V}_x =$  of x inside $\mathrm{B} =$ , such that $\mathcal{G}|_{\mathrm{V}_x}$ is constant. It is then easy to see

$\iota^!\mathcal{G}_x = \mathrm{Cone}\Big(\mathrm{R}\Gamma\big($  $; \mathbf{Q}^{\mathrm{rank}(\mathcal{E})}\big) \to \mathrm{R}\Gamma\big($  $; \mathbf{Q}^{\mathrm{rank}(\mathcal{E})}\big)\Big) = 0.$

For the point s, we also choose a sufficiently small neighborhood V_s which is the intersection of a small ball around s with B. In this situation, we have an exact sequence

$$
\cdots \to \mathrm{H}^m(\iota^!\mathcal{G}_s) \to \mathrm{H}^m(\mathrm{V}_x, \{s\}; \mathbf{Q}^{\mathrm{rank}(\mathcal{E})}) \to \mathrm{H}^m(\mathrm{V}_s - (\mathbf{S} \cap \mathrm{V}_s); \mathbf{Q}^{\mathrm{rank}(\mathcal{E})}) \to \cdots.
$$

The middle term is identically zero, and the right hand side is nonzero only if $m = 0$. This means that $\mathrm{H}^m(\iota^! \mathcal{G}_s) = 0$ only if $m = 1$.

Proof of (c). We have $\mathrm{H}^m(\mathbf{A}^1; i_* \mathrm{R} j^\circ_* j^{\circ *}(i^* j_! \mathcal{E})) = \mathrm{H}^m(\mathrm{B}^\circ; j_! \mathcal{E}|_{\mathrm{B}^\circ})$. Since B° is diffeomorphic to $\mathbf{A}^1$, since $\mathrm{S} \cap \mathrm{B}^\circ$ has fewer points than S, and since $(j_! \mathcal{E})_{\mathrm{B}^\circ}$ is the sheaf on B° obtained from the local system $\mathcal{E}|_{\mathrm{B}^\circ - (\mathrm{S} \cap \mathrm{B}^\circ)}$ by extension by zero, (c) can be deduced from the inductive hypothesis. This concludes the proof of Lemma 3.2. □

Notes

1. Theorem 2.4 for torsion sheaves in étale topology is given in SGA 4, Exposé XIV, Corollaire 3.5. Both Theorem 2.4 and its étale version are conventionally referred to as the Artin vanishing theorem. This theorem is a special case of a more general theorem in relative setting for an affine morphism. In our setting, this more general theorem reads:

Theorem. *Let* $f \colon \mathrm{X} \to \mathrm{Y}$ *be an affine morphism between complex algebraic varieties. Assume that* $\dim \mathrm{X} \le n$. *Let* $\mathcal{F}$ *be a constructible sheaf on* X. *Then*

$$\dim \operatorname{Supp} \mathrm{R}^q f_* \mathcal{F} \le n - q.$$

It is clear that when Y reduces to a point, the above theorem specializes to Theorem 2.4. Moreover, recalling Definition 2.6, arguing as in Proposition 2.8, we find the above theorem implies that $\mathrm{R} f_* ({}^{\mathrm{p}}\mathcal{D}^{\le 0}_{\mathrm{X}}) \subset {}^{\mathrm{p}}\mathcal{D}^{\le 0}_{\mathrm{Y}}$. This last result is also called the Artin vanishing theorem in some literature.

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