Lectures V & VI. Vanishing cycles

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1 Hypersurface singularities

In this lecture we introduce two functors called the nearby cycle functor and the vanishing cycle functor. These functors are defined by Grothendieck while studying Milnor's theory on singular hypersurfaces. We thereby begin by reviewing Milnor's construction of his fibration in the local situation.

1.1 Milnor fibration

Construction 1.1. Consider a holomorphic function $f\colon(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$. Choose $0<\epsilon\ll 1$ and $0<\eta\ll\epsilon$, such that f is defined in a neighborhood of an open ball B_ϵ of radius ϵ in $\mathbf{C}^{n+1}$, and, denote D_η the open disk in $\mathbf{C}$ of radius η. Form the restriction map

$$\boldsymbol{x}\mapsto f(\boldsymbol{x})\colon f^{-1}(D_\eta-\{0\})\cap B_\epsilon\to D_\eta-\{0\}.$$

We shall write $X=f^{-1}(D_\eta)\cap B_\epsilon$, $S=D_\eta$, $X_0=f^{-1}(0)\cap B_\epsilon$, $S^*=S-\{0\}$, and $X^*=X-X_0$. The maps $X\to S$, $X^*\to S^*$ induced by f will still be denoted by f.

Theorem 1.2 (Milnor 1968)**.** *In Construction 1.1, if ϵ and η are suitably chosen, then $f\colon X^*\to S^*$ is a locally topologically trivial fibration.*

Definition 1.3 (Milnor fibration, Milnor fiber)**.** In Construction 1.1, the holomorphic map $f\colon X^*\to S^*$ is called the (holomorphic) *Milnor fibration*. Let $F_{f,0}$, or simply F_f, be a typical fiber of the Milnor fibration. Then F_f is an n-dimensional complex manifold, and is called the *Milnor fiber* of the germ f. The intersection $X_0\cap\partial\overline{B}_{\epsilon'}$, where $0<\epsilon'<\epsilon$, is called the *link* of the singularity.

In literature, people often refer to a germ $f\colon(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$ having a critical point at $0\in\mathbf{C}^{n+1}$ as a *hypersurface singularity*, for the obvious reason. When 0 is an isolated critical point of f, we say f is an *isolated* hypersurface singularity.

Remark 1.4. The usage of terminology "hypersurface singularity" is not entirely accurate, since the singularity of the level hypersurface $f^{-1}(0)$ is only a part of the information provided by the germ f.

If f is already smooth at 0, then F_f is clearly contractible. In general, F_f is a geometric object whose complexity reflects the singularity of f near 0.

Exercise 1.5. Show that the germ $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ has an isolated critical point at 0 if and only if the *Milnor algebra*

$$\frac{\mathbf{C}[\![x_0,\dots,x_n]\!]}{\left(\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_n}\right)}$$

is a finite dimensional nonzero $\mathbf{C}$-algebra.

Example 1.6 (Morse singularity)**.** The most basic example of a germ with critical point is the *Morse singularity*, or A_1 singularity, given by $f(x_0,\dots,x_n) = x_0^2 + \cdots + x_n^2$. For this example, we can choose η and ϵ pretty arbitrarily. The Milnor fiber is then diffeomorphic to the level set $f = 1$. When $n = 1$, the link of the Morse singularity is the Hopf link in $\mathbf{S}^3$.

Exercise 1.7. Show that the Milnor fiber of the Morse singularity is homotopy equivalent to $\mathbf{S}^n$.

Example 1.8 (A_N singularity)**.** The n-dimensional A_N singularity is defined by the germ of the polynomial $f(x_0,\dots,x_n) = x_0^{N+1} + x_1^2 + \cdots + x_n^2$. When $N = n = 2$, the link of A_2 singularity is the trefoil knot in $\mathbf{S}^3$.

Example 1.9 (Sum of singularities)**.** If f_1 is an n_1 dimensional hypersurface singulairty, and f_2 is an n_2 dimensional hypersurface singulairty, then their *sum* is defined as the germ

$$(\boldsymbol{x}_1,\boldsymbol{x}_2) \mapsto f_1(\boldsymbol{x}_1) + f_2(\boldsymbol{x}_2)\colon (\mathbf{C}^{n_1+n_2+2},0) \to (\mathbf{C},0).$$

Thus, the A_N singularity defined in Example 1.8 is a sum of the 0-dimensional singularity $x \mapsto x^{N+1}$ with 0-dimensional Morse singularities.

Exercise 1.10. Show that if f_1 and f_2 are isolated hypersurface singularities, their sum is also an isolated hypersurface singularity.

Example 1.11 (Normal crossing singularity)**.** Let $e_0,\dots,e_n$ be non-negative integers. Then the germ of the monomial $f(x_0,\dots,x_n) = x_0^{e_0}\cdots x_n^{e_n}$ is called a *monomial singularity*, or a *normal crossing singularity.*

Monomial singularities hold paramount importance in singularity theory and algebraic geometry because of the following Main Theorem II′(N) of Hironaka 1964.

Theorem 1.12 (Hironaka)**.** *Let X be a complex manifold. Let Z be a closed analytic subspace. Then for any point z of Z, there is a neighborhood U of z in X, a projective holomorphic map $\pi\colon \widetilde{U} \to U$ that is the composition of a sequence of blowups with smooth centers, such that*

- $\pi|_{\pi^{-1}(U - U\cap Z)}$ *is an isomorphism between $U - U \cap Z$ and $\widetilde{U} - \pi^{-1}(Z)$,*
- $E = \pi^{-1}(Z)$ *is a union of nonsingular divisors: $E = \bigcup_{i\in I} E_i$,*
- *for any nonempty subset $J \subset I$, $\bigcap_{j\in J} E_j$ is a nonsingular closed submanifold of $\widetilde{U}$ with equal codimension* Card J,
- *the strict transform $\widetilde{Z}$ of Z is nonsingular.*

Suppose $f\colon X \to S$ is a representative of a hypersurface singularity. By applying Hironaka's theorem to the pair (X, X_0) (shrinking ϵ and η if necessary), we can find a projective morphism $\pi\colon \widetilde{X} \to X$, such that the $E = \pi^{-1}(X_0)$ is a divisor with normal crossing, The composition $g = f \circ \pi\colon \widetilde{X} \to S$ is zero precisely along E. Near every point a of E, we can choose a coordinate neighborhood Y of a with coordinates $y_0,\dots,y_n$ such that the coordinate of a is $(0,\dots,0)$, and the normal crossing divisor E is defined by $y_0\cdots y_r = 0$. In this situation, $g(y)$ is locally of the form $h(y)\cdot y_0^{e_0}\cdots y_r^{e_r}$, where $h(y)$ is nonzero along E. By changing the coordinate of S and shrinking, we can ensure $g\colon Y \to S$ is a representative of a monomial singularity. On other hand, since $\widetilde{X} - E$ is isomorphic to $X - f^{-1}(0)$, the fibration $\widetilde{X} - E \to S^*$ is equal to the Milnor fibration $X^* \to S^*$. This procedure tells us that the study of a Milnor fibration $f\colon X \to S$ can always be replaced by $g\colon \widetilde{X} \to S$, so that the singular fiber X_0 is replaced by a normal crossing filling. This paragraph will be the foundation of Grothendieck's sheaf-theoretic proof of the monodromy theorem.

Exercise 1.13. Show that in general a monomial singularity is not isolated. Show that the Milnor fiber of a monomial singularity is a homeomorphic to a disjoint union of products of $\mathbf{C}^*$.

1.2 Topology of Milnor fiber

The Milnor fiber F_f, being a nonsingular closed subspace in a Stein manifold, is also a Stein manifold. Therefore we have the following theorem.

Theorem 1.14. *If $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ is a hypersurface singularity, then $\mathrm{H}_i(F_f;\mathbf{Z}) = 0$ for $i > n$.*

Proof. The standard Morse theory argument (Milnor 1963, p. 39) works. See also Milnor 1968, Theorem 5.1. □

For an isolated hypersurface singularity, Milnor proves that the homotopy type of its Milnor fiber F_f is extremely simple. The key is the following.

Proposition 1.15. *Let $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ be an isolated hypersurface singularity. Then the reduced homology $\widetilde{\mathrm{H}}_i(F_f;\mathbf{Z})$ is trivial for any $i \neq n$, and $\widetilde{\mathrm{H}}_n(F_f;\mathbf{Z})$ is a finitely generated free abelian group.*

We shall prove Proposition 1.15 and its generalization for non-isolated hypersurface singularity later using a sheaf-theoretic method (see Proposition 2.20). It turns out that Proposition 1.15 and its generalization will be consequences that the "vanishing cycle functor", to be introduced shortly, preserves perverse sheaves.

Proposition 1.15 results the following homotopy theoretic result.

Theorem 1.16 (ibid., Theorem 6.5)**.** *In Construction 1.1, if 0 is an isolated critical point (i.e., $f\colon X \to S$ is smooth away from $0 \in X \subset \mathbf{C}^{n+1}$), then F_f is homotopy equivalent to a bouquet of μ spheres.*

Proof. The $n = 1$ case being easy, we shall assume that $n \geq 2$. In this case, it can be shown that F_f is simply connected (see, e.g., ibid., Theorem 5.2). Since the reduced homology of F_f is concentrated in degree n only, and is freely generated, Hurewicz implies $\pi_n(F_f) \simeq \mathbf{Z}^\mu$ for some natural number μ. This gives rise to a map from a bouquet of μ spheres into F_f. Moreover, it induces an isomorphism on all homology groups, thanks to Theorem 1.14 and Proposition 1.15. The theorem then follows from Whitehead's theorem (see, e.g., Hatcher 2002, Corollary 4.33). □

Definition 1.17. Assume that the germ f has an isolated critical point at 0. Then the number μ, which equals the dimension of $\widetilde{\mathrm{H}}_n(F_f;\mathbf{Q})$, is called the *Milnor number* of the germ f.

While there are quite fruitful differential-geometric study of F_f, for our limited purposes we shall be content with studying the linear algebra object $\mathrm{H}^*(F_f;\mathbf{Q})$. The immediate question is how to compute μ. Milnor proved the following theorem.

Theorem 1.18. *Assume that $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ is a germ of a holomorphic function having an isolated critical point at $0 \in \mathbf{C}^{n+1}$. Then we have*

$$\boxed{\mu = \dim_{\mathbf{Q}} \widetilde{\mathrm{H}}_n(F_f;\mathbf{Q}) = \dim_{\mathbf{C}} \frac{\mathbf{C}[\![x_0,\ldots,x_n]\!]}{\left(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right)}}. \tag{1.19}$$

The formula (1.19) is known as *Milnor's formula*. By Exercise 1.5 the displayed quotient ring is indeed an artinian local $\mathbf{C}$-algebra, hence is finite dimensional.

We shall provide an algebraic proof of Milnor's formula, using global methods.

Example 1.20. In view of formula (1.19), the Milnor number of the sum of f_1 and f_2 is equal to the product $\mu_{f_1} \cdot \mu_{f_2}$ of their respective Milnor numbers. In particular, the Milnor number of the A_N singularity is equal to N.

1.3 Monodromy of Milnor fiber

Next, let me turn to another important discrete invariant that we can attach to the hypersurface singularity f, its *monodromy*.

Construction 1.21 (Monodromy automorphism)**.** The Milnor fiber F_f underlies an automorphism (up to isotopy) called the *monodromy automorphism*. Let $\gamma\colon [0,1] \to D$ be a loop going around the center $0 \in D$ counterclockwise. Then $X \times_{D,\gamma} [0,1]$ is a locally topologically trivial fibration over the closed interval $[0,1]$. By Ehresmann's method, the vector field $\frac{\mathrm{d}}{\mathrm{d}t}$ on $[0,1]$ generates a vector field on X which does not escape to infinity (upon shrinking ϵ and η). This vector field integrates a flow and gives rise to an automorphism of $F_f = f^{-1}(\gamma(0)) = f^{-1}(\gamma(1))$. This is called the monodromy automorphism of F_f. The monodromy automorphism then induces the *monodromy operator* T on $\mathrm{H}^*(F_f)$.

Example 1.22 (Quasi-homogeneous singularity)**.** Let $f(x_0, \dots, x_n)$ be a polynomial. We say f is *quasi-homogeneous*, if there exist rational numbers $w_0, w_1, \dots, w_n$, called *weights* of the repective variables $x_0, \dots, x_n$, such that for any $t \in \mathbf{C}$, we have

$$tf(x_0, \dots, x_n) = f(t^{w_0}x_0, \dots, t^{w_n}x_n).$$

For example, the A_N singularity $f(x_0, \dots, x_n) = x_0^{N+1} + x_1^2 + \cdots + x_n^2$ (Example 1.8) is quasi-homogeneous with weights $w_0 = \frac{1}{N+1}$, and $w_i = 1/2$ for $i > 0$. Every homogeneous polynomial of degree d is quasi-homogeneous of weights $w_i = 1/d$. More generally, the Brieskorn–Phạm singularity $f(x_0, \dots, x_n) = x_0^{k_0} + \cdots + x_n^{k_n}$ $(k_i \geq 1)$ is an isolated quasi-homogeneous singularity.

Let us prove a quasi-homogeneous singularity f has finite monodromy using a specific example $f(x,y) = x^2 + y^3$. The Milnor fiber in this case is diffeomorphic to the level hypersurface $f = 1$. In fact, all level sets $f = t$, $t \neq 0$, are isomorphic: $(x,y) \mapsto (xt^{-1/2}, yt^{-1/3})$ establishes the isomorphism between $f = t$ and $f = 1$. However there are some ambiguity (extra choice) of the isomorphy, in that we have to choose a square root and a cubic root of t. The ambiguity is determined by the group $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3$. Thus, letting $T\colon \mathbf{C}^*_\tau \to S^* = \mathbf{C}^*_t$ be the map $\tau \mapsto \tau^6$, we have $X^* \times_{S^*} T$ is a trivial family, isomorphic to $\{f = 1\} \times T$. This proves the monodromy is finite.

Remark 1.23. As shown by Brieskorn 1966, when $n = 4$, the links of the Brieskorn–Phạm singularity with $k_0 = k_1 = k_2 = 2$, $k_3 = 3$, and $k_4 = 6k - 1$ for $k = 1, 2, \dots, 28$ gives all 28 possible smooth structures on the oriented 7-sphere. The links of some higher dimensional Brieskorn–Phạm singularities also give higher dimensional exotic spheres.

Motivated by Example 1.22, Milnor formulated a conjecture asserting that for any holomorphic germ f, the eigenvalues of T on $\mathrm{H}^*(F_f; \mathbf{Q})$ are always roots of unity.

Remark 1.24. In fact, Milnor even asked whether it is true that $T|_{\mathrm{H}^*(F_f;\mathbf{Q})}$ is always semisimple (at least for germs with an isolated critical point at 0). This turns out to be too much to ask for. The simplest isolated singularity with non-semisimple monodromy is $f(x,y) = x^5 + y^5 + x^2y^2$ around $(0,0) \in \mathbf{C}^2$, as can be computed by Singular (see Remark 1.30).

Milnor's conjecture is subsequently proved by Grothendieck using Hironaka's resolution of singularity.

Theorem 1.25 (Local monodromy theorem, version I)**.** *Let $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a hypersurface singularity. Then for any $m \geq 0$, the monodromy $\mathbf{Z}(1)$-action on $\mathrm{H}^m(F_f; \mathbf{Z})$ is quasi-unipotent, i.e., the eigenvalues any $\gamma \in \mathbf{Z}(1)$ are roots of unity. Moreover, the size of any Jordan block of $\gamma|_{\mathrm{H}^m(F_f;\mathbf{Z})}$ does not exceed $m+1$.*

Grothendieck's proof works in quite general context and uses a new tool called the vanishing cycle functor. Grothendieck's vanishing cycle functor turns out to be closely related to perverse sheaves and microlocal geometry. I hope to illustrate this point step by step.

The local monodromy theorem stated above also has a variant for families of varieties over a punctured disk.

Theorem 1.26 (Local monodromy theorem, version II)**.** *Let X be a complex manifold. Let S be a small disk in $\mathbf{C}$. Let $f\colon X \to S$ be a proper map which is smooth away from $f^{-1}(0)$. Then for any nonzero t in S, and any $m \geq 0$, the monodromy $\mathbf{Z}(1)$-action on $\mathrm{H}^m(f^{-1}(t); \mathbf{Z})$ is quasi-unipotent. Moreover, for any $\gamma \in \mathbf{Z}(1)$, the size of any Jordan block of $\gamma|_{\mathrm{H}^m(f^{-1}(t);\mathbf{Z})}$, does not exceed $m+1$.*

Remark 1.27. In Theorem 1.26, it suffices to assume f is a *stratified submersion* away from special fiber. Also, the properness can also be dropped if a certain finiteness condition is imposed (e.g., if X is a family of algebraic varieties over S).

For instance, suppose $f\colon X \to S$ can be compactified as $\overline{f}\colon \overline{X} \to S$, such that $\overline{f}$ is proper, $\overline{X}$ is a complex manifold, and $\overline{X} - X$ is a relative normal crossing divisor when restricted to S^*. Then the quasi-unipotence of $\mathrm{H}^m_c(f^{-1}(t); \mathbf{Z})$, hence by Poincaré duality the quasi-unipotence of $\mathrm{H}^m(f^{-1}(t); \mathbf{Z})$, follows from the quasi-unipotence of $\mathrm{H}^m(\overline{f}^{-1}(t); \mathbf{Z})$ and that of $\mathrm{H}^m(\overline{f}^{-1}(t) - f^{-1}(t); \mathbf{Z})$. The first case has been covered by Theorem 1.26; the second case follows from an argument using a Mayer–Vietoris type spectral sequence.

Using "cohomological descent for proper hypercoverings", one can then deduce the quasi-unipotence for singular compactifiable spaces over S. We omit the details. The reader who is unfamiliar with this formalism should turn to Deligne 1971; Deligne 1974 for some cultural training.

Example 1.28 (Tate's elliptic curve)**.** Let $\Delta^* = \{q \in \mathbf{C} : 0 < |q| < 1\}$ be the unit punctured disk with coordinate q. Define $X(q) = \mathbf{C}^*/q^{\mathbf{Z}}$. This is a compact complex manifold diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$. When q varies in Δ^*, the spaces $\{X(q)\}_{q \in \Delta^*}$ form a "family of elliptic curves" over Δ^*, i.e., there is a complex manifold X^* and a proper holomorphic map $f\colon X^* \to \Delta^*$ such that $f^{-1}(q) = X(q)$. Moreover, this family can be "compactified", in that there is a complex manifold X, containing X^* as an open subspace, a proper holomorphic map $X \to \Delta$, still denoted by f, such that the following diagram

$$\begin{array}{ccc} X^* & \hookrightarrow & X \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^* & \hookrightarrow & \Delta \end{array}$$

is cartesian.

For each $q \in \Delta^*$, the homology of $X(q)$ is free of rank 2, with two generators a and b chosen as follows: a is the class of the unit counter-clockwise circle in $\mathbf{C}^*$, b is the image of the path joining 1 and $q \in \mathbf{C}^*$. The cycle b is the so-called "vanishing homology class" of this degeneration; it shrinks to a point as $q \to 0$. Let us examine the monodromy action on the homology space $\mathbf{Z}^2$. Start with, say, $q = 1/2$, and let q wind around 0 counterclockwise. The cycle $b(q)$ is homologous to the cycle $b(0)$ followed by an arc of the circle $|q| = 1/2$. When q goes a full circle, we find $b(0)$ becomes homologous to $b(0) + a$. Thus the monodromy on homology is given by the integral matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

with respect to the basis a, b of the homology. It is unipotent with Jordan block of size 2.

Remark 1.29*.* Depending on the context, you may find this theorem to be attributed to other authors such as Brieskorn, Borel, or Landman. The history is not that convoluted, however. Grothendieck is the first who proved both theorems, by inventing the vanishing cycle functor. Brieskorn 1970 proved Theorem 1.25 (no bounds on sizes of Jordan blocks) for isolated hypersurface singularities by inventing a de Rham theoretic approach to singularities, and by beautifully applying Gelfond's theorem stating that α and $\mathrm{e}^{2\pi\mathrm{i}\alpha}$ can't be both algebraic unless α is rational. Brieskorn's method also works Theorem 1.26 (see Deligne 1970, III, §2).

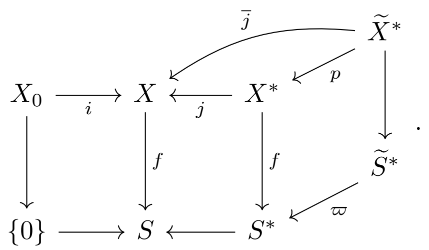
Borel proved a quasi-unipotence result for a wider range of local systems on a punctured disk, namely those underlie rational variation of Hodge structure, thus implies Theorem 1.26 (but not Theorem 1.25). Landman in his thesis gave a direct geometric flavored proof of Theorem 1.26. Katz's influential 1970 paper gives an "arithmetic" proof of Theorem 1.26 when f comes from a family of algebraic varieties. It also has an improved upper bound on the sizes of Jordan blocks in terms of the Hodge levels.

Remark 1.30*.* Grothendieck's sheaf-theoretic approach however has a drawback: it is not suitable for explicit computation. For example, while it offers a bound of the sizes of the Jordan blocks of T, it is not clear how to determine the sizes when the power series expansion of f is given. A more effective, de Rham theoretic, approach is developed by Brieskorn (1970) for isolated singularities, which in turn influences the development of $\mathcal{D}$-Modules through the work of Malgrange and Kashiwara. Nowadays, Brieskorn's method has been incorporated into the computer program Singular by Schulze, through the library `gmssing`. I know I am making a strong statement, but for an isolated hypersurface singularity, `gmssing` can calculate all the discrete invariants that you can attach to the singularity for you.

2 Nearby cycle and vanishing cycle

2.1 The nearby cycle functor

Situation 2.1. Let X be a complex manifold. Let $S = \mathbf{A}^{1,\mathrm{an}}$. Denote by S^* the open submanifold $S - \{0\}$ of S, and $\widetilde{S}^*$ the universal cover of S^*. Let $f\colon X \to S$ a holomorphic function. Then we can form the following commutative diagram, in which all squares are cartesian:



The group of deck transformations of the covering map $\widetilde{S}^* \to S^*$ is canonically isomorphic to $\mathbf{Z}(1) = 2\pi\mathrm{i}\mathbf{Z}$; if $z \in \widetilde{S}$ and $\gamma \in \mathbf{Z}(1)$, then the deck transformation is given by $z \mapsto z + \gamma$.

Definition 2.2. (1) For a complex of sheaves of abelian groups $\mathcal{F}$ on X, the *nearby cycle* $R\Psi_f(\mathcal{F})$ is defined by

$$\boxed{R\Psi_f(\mathcal{F}) = i^* R\overline{j}_* \overline{j}^* \mathcal{F}}.$$

This is a complex of abelian sheaves on X_0. Note that immediately from the definition we have $R\Psi_f(\mathcal{F})$ only depends on $\mathcal{F}|_{X^*}$, and does not care about the behavior of the $*$- or !-restriction of $\mathcal{F}$ to X_0.

(2) Any deck transformation $\gamma\colon \widetilde{S}^* \to \widetilde{S}^*$ induces a deck transformation, still denoted by γ, on $\widetilde{X}^*$. Since $\overline{j} \circ \gamma = \gamma$, for any sheaf $\mathcal{F}$ on X^*, we have a canonical identification

$$\gamma^* \overline{j}^* \mathcal{F} \xrightarrow{\sim} \overline{j}^* \mathcal{F},$$

which by adjunction gives

$$\overline{j}^* \mathcal{F} \to R\gamma_* \overline{j}^*_* \mathcal{F}.$$

Applying $i^* R\overline{j}_*$ to the above gives an endomorphism

$$\boxed{T_\gamma\colon R\Psi_f(\mathcal{F}) \to R\Psi_f(\mathcal{F})}$$

This action is called the *monodromy action.*

(3) There is also a canonical *specialization map*

$$\alpha\colon i^* \mathcal{F} \to \Psi_f(\mathcal{F})$$

which is obtained from the adjunction map $\mathcal{F} \to R\overline{j}_* \overline{j}^* \mathcal{F}$ by applying i^*.

Exercise 2.3. Check that for any $\gamma \in \mathbf{Z}(1)$, we have $T_\gamma \circ \alpha = \alpha$. In other words, the image of the specialization map is monodromy invariant.

Remark 2.4. The formal definition of nearby cycle functor first appeared in SGA 7_{I}, Exposé I, 2.2 for a constant sheaf, and then in SGA 7_{II}, Exposé XV in general. It was initially called "vanishing cycle functor" (*foncteur cycles évanescents*). Later, this name is attached (more approriately) to a related functor $R\Phi_f$, and the functor $R\Psi_f$ is renamed to the "nearby cycle functor" (*foncteur cycles proches*).

Remark 2.5. When we talk about the nearby cycle of a complex, we mean both the above complex and the monodromy action. Thus we should view $R\Psi_f(\mathcal{F})$ as an object in the derived category of complexes of sheaves of $\mathbf{Z}[\mathbf{Z}(1)]$-modules. Whenever we fix a generator T of $\mathbf{Z}(1)$, we may identify the group ring $\mathbf{Z}[\mathbf{Z}(1)]$ with the ring $\mathbf{Z}[T, T^{-1}]$ of Laurent polynomials.

Remark 2.6. For the sake of computation, we will frequently identify $\widetilde{S}$ with $\mathbf{C}$, and identify the covering map $\widetilde{S}^* \to S^*$ with $z \mapsto \exp(2\pi\mathrm{i}z)$. Under this identification, we also obtain an isomorphism $\mathbf{Z}(1) \cong \mathbf{Z}$, and the a positive generator of the deck transformation is given by $z \mapsto z + 1$.

Exercise 2.7. Let $t\colon S \to \mathbf{A}^1$ be a coordinate function of the disk S. Let $\mathcal{F}$ be a locally constant sheaf on S^*, corresponding to a $\mathbf{Z}[\mathbf{Z}(1)]$-module M. Show that $R\Psi_t(\mathcal{F})$ is a sheaf on a point, i.e., an abelian group. Show also that $R\Psi_t(\mathcal{F})$, equipped with the monodromy representation defined above, is isomorphic to M as $\mathbf{Z}[\mathbf{Z}(1)]$-modules.

Proposition 2.8. *Let $f\colon X \to \mathbf{A}^1$ be a holomorphic function on a complex manifold. For any $x \in f^{-1}(0)$, let $F_{f,x}$ be the Milnor fiber of f at x. Then we have an isomorphism*

$$R^m\Psi_f(\mathbf{Z})_x \cong \mathrm{H}^m(F_{f,x};\mathbf{Z})$$

compatible with monodromy.

Proof. The situation being local, we might as well limit ourselves to a Milnor fibration $f\colon X \to S$. Let $\widetilde{S}^*_R$ be the open subspace of $\widetilde{S}^*$ consisting of complex numbers whose imaginary part is bigger than R, and $\widetilde{X}^*_R = \widetilde{X}^* \times \widetilde{S}_R$. Since $\widetilde{S}^*_R$ is contractible, $\widetilde{X}^*_R$ is isomorphic to $F_{f,x} \times \widetilde{S}^*_R$. Thus

$$R\Psi_f(\mathbf{Z})_x = \operatorname*{colim}_{R\to+\infty} R\Gamma(\widetilde{X}^*_R;\mathbf{Z}) = \operatorname*{colim}_{R\to+\infty} R\Gamma(F_{f,x}\times\widetilde{S}^*_R;\mathbf{Z}).$$

Since $\widetilde{S}^*$ is contractible, the right hand side is a constant system on $R\Gamma(F_{f,x};\mathbf{Z})$. Unwinding the definition, the monodromy actions on both sides agree. □

Proposition 2.9. *Let $\pi\colon Y \to X$ be a proper holomorphic map of complex analytic spaces. Let π_0 be the induced map between $\pi^{-1}(f^{-1}(0))$ and $f^{-1}(0)$. Let $f\colon X \to \mathbf{A}^1$ be a holomorphic function. Then for any $\mathcal{F} \in D^+(Y)$, we have a natural isomorphism*

$$R\Psi_f(R\pi_*\mathcal{F}) \xrightarrow{\sim} R\pi_{0*}R\Psi_{f\circ\pi}(\mathcal{F}).$$

Proof. This is an exercise of the proper base change theorem. □

Corollary 2.10. *Let $f\colon X \to \mathbf{A}^1$ be a proper morphism between complex analytic spaces. Then we have an isomorphism*

$$\mathrm{H}^*(f^{-1}(0);R\Psi_f\mathbf{Z}_X) \approx \mathrm{H}^*(X_{\overline{\eta}};\mathbf{Z}).$$

Here, $\overline{\eta}$ is any point of $\mathbf{A}^1$ sufficiently close to 0. Moreover, the isomorphism is compatible with the monodromy actions.

Proof. Let t be the coordinate function of $\mathbf{A}^1$. Let S be a small disk around 0 such that $Rf_*\mathbf{Z}_X$ is lisse on the punctured S^*. According to the description of nearby cycle functor on a 1-dimensional base (Exercise 2.7), we know that $R\Psi_t(\mathcal{F}) = \mathcal{F}_{\overline{\eta}}$ for any $\overline{\eta} \in S^*$, and the monodromy actions on both sides agree. It follows from Proposition 2.9 (with $\pi = f$, $f = t$) that

$$R\Gamma(f^{-1}(0);R\Psi_f\mathbf{Z}_X) \approx R\Psi_t(Rf_*\mathbf{Z}_X).$$

This completes the proof. □

Remark 2.11. Proposition 2.9 is not true is the properness cannot be dropped. For example, consider the function $f(x,y) = xy\colon \mathbf{A}^2 - \{0\} \to \mathbf{A}^1$. Note that the domain is $\mathbf{A}^2 - \{0\}$, which is obtained from $\mathbf{A}^2$ by removing the critical point 0. Since f is everywhere smooth, $R\Psi_f\mathbf{Z}_X$ is the constant sheaf, and we have $R\Gamma(f^{-1}(0);R\Psi_f\mathbf{Z}_X) \simeq \mathbf{Z}^2[0] \oplus \mathbf{Z}^2[-1]$, but the cohomology of a nearby fiber is given by $\mathbf{Z}[0]\oplus\mathbf{Z}[-1]$.

Example 2.12 (Nearby cycle for monomials)**.** Suppose X is a small open polydisk neighborhood of $0 \in \mathbf{C}^n$, f is given by $(x_1,\dots,x_n) \mapsto x_1^{e_1}\cdots x_r^{e_r}$, and $\mathcal{F}$ is the constant sheaf $\mathbf{Z}_X$. Then the stalk of the nearby cycle functor at $0 \in X$ is

$$\operatorname*{colim}_{\substack{\epsilon\to 0\\ R\to+\infty}} R\Gamma(X_R,\mathbf{Z}),$$

where $X_{\epsilon,R} = \{(x,z) \in X\times\mathbf{C} : |x| < \epsilon, f(x) = \exp(2\pi \mathrm{i} z), \operatorname{Im} z > R\}$.

Note that $X_{\epsilon,R}$ is homotopy equivalent to the space $F = \{x \in (\mathbf{S}^1)^n : x_1^{e_1}\cdots x_r^{e_r} = 1\}$. Hence the nearby cycle is computing the cohomology of F. Let e be the greatest common divisor of $e_1,\dots,e_r$. Then

F is homeomorphic to a disjoint union of the same $F = \bigsqcup_{i=0}^{e-1} F_i$, where F_i are all homeomorphic to the $(r-1)$-dimensional torus F_0 defined by the exact sequence

$$1 \to F_0 \to (\mathbf{S}^1)^r \xrightarrow{\phi} \mathbf{S}^1 \to 1, \quad \phi(x_1, \ldots, x_n) = (x_1^{e_1} \cdots x_r^{e_r})^{1/e}.$$

It follows that

$$\begin{aligned} R^q\Psi_f(\mathbf{Z})_0 &= \mathbf{Z}[\mathbf{Z}/e\mathbf{Z}] \otimes \mathrm{H}^q(F_0, \mathbf{Z}), \\ \mathrm{H}^q(F_0, \mathbf{Z}) &= \textstyle\bigwedge^q \mathrm{H}^1(F_0, \mathbf{Z}), \\ \mathrm{H}^1(F_0, \mathbf{Z}) &= \mathrm{Coker}(\mathbf{Z} \to \mathbf{Z}^r, 1 \mapsto (e_i e^{-1})). \end{aligned}$$

Moreover, the monodromy action is given by the cyclotomic character $\mathbf{Z}(1) \to \boldsymbol{\mu}_e$. If we choose a primitive e^{th} root of unity ζ, then the monodromy action is given by the automorphism of F defined by $(x_1, \ldots, x_n) \mapsto (\zeta x_1, \ldots, \zeta x_r, x_{r+1}, \ldots, x_n)$.

2.2 Proof of the local monodromy theorem

With the above preparations ready, we are now in the position to prove the two versions of the local monodromy theorem.

Proof of Theorem 1.25. Let $f \colon X \to S$ be the Milnor fibration associated to the germ f. Let $\pi \colon Y \to X$ be an embedded resolution of $f^{-1}(0)$, so that the preimage $D = \pi^{-1}(0)$ is a normal crossing divisor in Y (see Theorem 1.12 and the discussion that follows it). Since π is an isomorphism away from $f^{-1}(0)$ and $g^{-1}(0)$, we have by Proposition 2.9 that $R\Psi_f(R\pi_*\mathbf{Z}_Y) = R\pi_{0*}R\Psi_g(\mathbf{Z})$. Moreover, since $(R\pi_*\mathbf{Z}_Y)|_{X^*} = \mathbf{Z}_{X^*}$, we have $R\Psi_f(R\pi_*(\mathbf{Z}_Y)) = R\Psi_f(\mathbf{Z}_X)$. By proper base change, and Proposition 2.8, we have

$$\begin{aligned} R\Gamma(F_f; \mathbf{Z}) &= R\Psi_f(\mathbf{Z}_X)_0 \\ &= R\pi_{0*}(R\Psi_g(\mathbf{Z}_Y))_0 \\ &= R\Gamma(D; R\Psi_g(\mathbf{Z}_Y)). \end{aligned}$$

Since g is a monomial function near any point of D, we can use Example 2.12 to conclude that the monodromy action of $R^m\Psi_g(\mathbf{Z}_Y)$ is semisimple stalk-by-stalk, whose eigenvalues are roots of unity. Using the spectral sequence of $\mathbf{Z}[\mathbf{Z}(1)]$-modules,

$$E_2^{p,q} = \mathrm{H}^p(D; R^q\Psi_g(\mathbf{Z}_Y)) \Rightarrow \mathrm{H}^{p+q}(D; R\Psi_g(\mathbf{Z}_Y)) = \mathrm{H}^{p+q}(F_f; \mathbf{Z}).$$

we see immediately that the eigenvalues of $\mathrm{H}^m(F_f; \mathbf{Z})$ are roots of unity as well. To get the bound on the sizes of Jordan blocks, we note that for fixed m, there are $m+1$ terms in the E_2 page of this spectral sequence that can contribute to $\mathrm{H}^m(D; R\Psi_g(\mathbf{Z}_Y)) = \mathrm{H}^m(F_f; \mathbf{Z})$. Thus $\mathrm{H}^m(F_f; \mathbf{Z})$ is a subquotient of at most $m+1$ semisimple $\mathbf{Z}(1)$-modules, and it follows that the sizes of Jordan blocks of $\mathrm{H}^m(F_f; \mathbf{Z})$ are at most $m+1$. □

Proof of Theorem 1.26. By Corollary 2.10, we have $\mathrm{H}^m(X_{\overline{\eta}}; \mathbf{Z}) \simeq \mathrm{H}^m(X_0; R\Psi_f\mathbf{Z})$ as $\mathbf{Z}[\mathbf{Z}(1)]$-modules. We have an spectral sequence of $\mathbf{Z}[\mathbf{Z}(1)]$-modules

$$\mathrm{H}^p(X_0; R^q\Psi_f\mathbf{Z}_X) \Rightarrow \mathrm{H}^{p+q}(X_0; R\Psi_f\mathbf{Z}) = \mathrm{H}^{p+q}(X_{\overline{\eta}}; \mathbf{Z}).$$

By Example 1.11, the monodromy action on any stalk of $R^q\Psi_f\mathbf{Z}_X$ is of finite order. Since f is proper, there exist a sufficiently large N, such that the monodromy $\mathbf{Z}[\mathbf{Z}(1)]$-action on $R^q\Psi_f\mathbf{Z}_X$ satisfies $\gamma^N = 1$ for any $\gamma \in \mathbf{Z}(1)$. It follows that the monodromy actions on the abutments spectral sequence are all of finite order. Since $\mathrm{H}^m(X_{\overline{\eta}}; \mathbf{Z})$ is a subquotient of an iterated extensions of $\mathbf{Z}[\mathbf{Z}(1)]$-modules of finite order, it is quasi-unipotent. Since $\mathrm{H}^m(X_{\overline{\eta}}; \mathbf{Z})$ has a composition series of length at most $i+1$, the sizes of its Jordan blocks do not exceed $m+1$. This completes the proof. □

Remark 2.13 (Arithmetic analogue of the local monodromy theorem)*.* Theorem 1.26 has an arithmetic analogue for any scheme defined over a henselian trait; the singular cohomology has to be replaced by ℓ-adic cohomology, where ℓ is a prime different from the residue characteristic of the trait. Thus let R be a henselian discrete valuation ring, $\{0\} \to \operatorname{Spec} R$ the inclusion of the spectrum of the residue field ("special point"), $\{\eta\} \to \operatorname{Spec} R$ the inclusion of the spectrum of the field of fractions of R ("generic point"), and let $\overline{\eta}$ be a separable closure of η.

Now let X be a scheme of finite type over $\operatorname{Spec} R$, and write its geometric generic fiber as $X_{\overline{\eta}}$, i.e., $X_{\overline{\eta}} = X \times_R \overline{\eta}$. Then the arithmetic analogue local monodromy theorem, Theorem 1.26, says the following. There is a finite indexed subgroup I' of the inertia group $I \subset \operatorname{Gal}(\overline{\eta}/\eta)$, such that the Galois representations $\rho\colon I' \to \operatorname{GL}(\mathrm{H}^m(X_{\overline{\eta}}; \mathbf{Z}_\ell))$ and $\rho_c\colon I' \to \operatorname{GL}(\mathrm{H}^m_c(X_{\overline{\eta}}; \mathbf{Z}_\ell))$ are unipotent.

The proof outlined above in the "geometric situation" carries over, since one can use alteration in place of embedded resolution of singularities. The details are reproduced in Proposition 6.3.2 of Berthelot's Bourkaki report.

2.3 Nearby cycle and perverse sheaves

Consider a holomorphic function $f\colon X \to \mathbf{A}^1$. For any critical point x of f such that $f(x) = 0$, we have shown that the Milnor fiber $F_{f,x}$ is closely related to the nearby cycle functor $R\Psi_f(\mathbf{Z}_X)$. Now I would like to indicate how to obtain some further results, especially I want to improve the vanishing result Proposition 1.15 and generalize it to non-isolated singularities. For this purpose I have to introduce the *vanishing cycle functor.*

Definition 2.14. In Situation 2.1, let $\mathcal{F}$ be an object of $D^+(X)$. We define the *vanishing cycle* of $\mathcal{F}$, denoted by $R\Phi_f(\mathcal{F})$, as the cone (or homotopy cokernel, or cofiber) of the specialization map $\alpha\colon i^*\mathcal{F} \to R\Psi_f(\mathcal{F})$ (Definition 2.2(3)). Thus, by definition, we have a distinguished triangle

$$\begin{array}{ccc} i^*\mathcal{F} & \xrightarrow{\alpha} & R\Psi_f(\mathcal{F}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{can}} \\ 0 & \longrightarrow & R\Phi_f(\mathcal{F}). \end{array}$$

in $D^+(X)$. This triangle is called the *canonical triangle.*

Remark 2.15. Definition 2.14 is not really legit, since the mapping cone of a map is not functorial in a triangulated category. There are multiple ways to mitigate this defect. For instance, one may define $\mathcal{K}$ as the complex

$$\overline{\varpi}_! \mathbf{Z}_{\widetilde{S}} \to \underset{\bullet}{\mathbf{Z}_{S^*}}$$

on S, where $\overline{\varpi}$ is the composition $\widetilde{S}^* \to S^* \to S$, and set

$$R\Phi_f(\mathcal{F}[-1]) = i^* R\mathcal{H}om(f^*\mathcal{K}, \mathcal{F}).$$

From the definition and Proposition 2.8 it is immediately clear that the stalk of $R\Phi_f(\mathbf{Z}_X)$ at a critical point $x \in f^{-1}(0)$ is isomorphic to the reduced cohomology of the Milnor fiber $F_{f,x}$. Thus, it follows that the support of $R\Phi_f(\mathbf{Z}_X)$ is completely contained in the critical locus of f along $f(0)$:

$$\boxed{\operatorname{Supp} R\Phi_f(\mathbf{Z}_X) \subset \operatorname{Crit}(f) \cap f^{-1}(0)}. \tag{2.16}$$

Thus, in contrast with the nearby cycle, the vanishing cycle has much smaller support. This feature makes it easier to work with vanishing cycles.

Nearby cycle and vanishing cycle functors work surprisingly well with perverse sheaves. In the following theorem, the coefficient ring of sheaves can be taken to be $\mathbf{Z}$.

Theorem 2.17. *Let $f\colon X \to \mathbf{A}^1$ be a regular function of algebraic varieties. Let $X_0 = f^{-1}(0)$. Then we have*

- $R\Psi_f[-1]({}^{\mathrm{p}}\mathcal{D}^{\leq 0}_X) \subset {}^{\mathrm{p}}\mathcal{D}^{\leq 0}_{X_0}$,
- $R\Phi_f[-1]({}^{\mathrm{p}}\mathcal{D}^{\leq 0}_X) \subset {}^{\mathrm{p}}\mathcal{D}^{\leq 0}_{X_0}$,
- $R\Psi_f[-1]({}^{\mathrm{p}}\mathcal{D}^{\geq 0}_X) \subset {}^{\mathrm{p}}\mathcal{D}^{\geq 0}_{X_0}$,
- $R\Phi_f[-1]({}^{\mathrm{p}}\mathcal{D}^{\geq 0}_X) \subset {}^{\mathrm{p}}\mathcal{D}^{\geq 0}_{X_0}$.

In particular, if $\mathcal{P}$ is a perverse sheaf on X, $R\Psi_f(\mathcal{P}[-1])$ and $R\Phi_f(\mathcal{P}[-1])$ are perverse sheaves on X_0.

Remark 2.18. It is common in literature to use ${}^{\mathrm{p}}R\Psi_f$ and ${}^{\mathrm{p}}R\Phi_f$ to denote $R\Psi_f[-1]$ and $R\Phi_f[-1]$.

Remark 2.19. There exists a suitable notion of stratification is for complex manifolds and complex analytic spaces. We can talk about constructibility and perversity of sheaves on complex manifolds and complex analytic spaces. With these suitable definitions, Theorem 2.17 is also true for complex analytic spaces.

We shall defer the proof of Theorem 2.17 since we have not developed the necessary homological algebra machinery. For now, let us exploit how it is related to the vanishing of Milnor cohomology.

Proposition 2.20. *Let $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a hypersurface singularity. Let ϵ be the dimension of the critical locus of f at 0. Then $\widetilde{\mathrm{H}}^m(F_f; \mathbf{Z}) = 0$ for $m < n - \epsilon$.*

Proof. This proof require some faith of the reader because we shall use the general notion of constructibility for complex manifolds mentioned in Remark 2.19 to define perverse sheaves on complex manifolds. Hence as we would anticipate, if $f\colon X \to S$ is the Milnor fibration, $\mathbf{Z}_X[n+1]$ is a perverse sheaf on X, and Theorem 2.17, tells us that $R\Phi_f(\mathbf{Z}[n])$ is a perverse sheaf on the analytic space $X_0 = f^{-1}(0)$, supported on the critical locus (2.16). We showed before that if $\mathcal{P}$ is a perverse sheaf on an ϵ-dimensional space, then its cohomology sheaves $\mathcal{H}^m\mathcal{P}$ are nonzero only when $m \in [-\epsilon, 0]$. This means that

$$\widetilde{\mathrm{H}}^{n-j}(F_f; \mathbf{Z}) = R^{-j}\Phi_f(\mathbf{Z}_X[n])_0$$

is nonzero only if $j \in [0, \epsilon]$. This completes the proof. □

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