LECTURE 7. BETTI NUMBER OF SINGULAR PROJECTIVE HYPERSURFACES

1. Nonsingular hypersurfaces

Let me begin with an exercise that everyone should have done once in their life, namely computing the Betti numbers of a nonsingular degree d hypersurface in $\mathbb{P}^n$.

Lemma 1.1. *Let V be a degree d, possibly singular hypersurface in $\mathbb{P}^n$. Then the restriction map $\mathrm{H}^i(\mathbb{P}^n;\mathbb{Q}) \to \mathrm{H}^i(V;\mathbb{Q})$ is an isomorphism for $i < n-1$.*

Proof. The complement $\mathbb{P}^n \setminus V$ is a nonsingular affine variety of pure dimension n. Artin's vanishing implies that $\mathrm{H}^i_c(\mathbb{P}^n \setminus V;\mathbb{Q}) = 0$ for all $i < n$. The result follows from the long exact sequence

$$\cdots \to \mathrm{H}^i_c(\mathbb{P}^n \setminus V;\mathbb{Q}) \to \mathrm{H}^i(\mathbb{P}^n;\mathbb{Q}) \to \mathrm{H}^i(V;\mathbb{Q}) \to \mathrm{H}^{i+1}_c(\mathbb{P}^n \setminus V;\mathbb{Q}) \to \cdots. \qquad \square$$

Corollary 1.2. *If V is a* nonsingular *degree d hypersurface in $\mathbb{P}^n$, then $\mathrm{H}^i(V;\mathbb{Q}) \simeq \mathrm{H}^i(\mathbb{P}^n;\mathbb{Q})$ for all $i \neq n-1$.*

Proof. This follows from Lemma 1.1 and Poincaré duality. $\square$

Therefore, to compute the dimension of $\mathrm{H}^{n-1}(V;\mathbb{Q})$ it suffices to know the Euler characteristic

$$\chi(V) = \sum (-1)^i \dim \mathrm{H}^i(V;\mathbb{Q}).$$

The Euler characteristic can be calculated by means of the Gauss–Bonnet theorem, which asserts that

$$\chi(V) = \int_V c_{n-1}(\Theta_V),$$

where Θ_V is the tangent bundle of V. We have an exact sequence

$$0 \to \Theta_V \to \Theta_{\mathbb{P}^n}|_V \to N_{V/\mathbb{P}^n} \to 0.$$

By the so-called adjunction formula, the normal bundle of V in $\mathbb{P}^n$ is equal to $\mathcal{O}_V(d)$, defined as the restriction of $\mathcal{O}_{\mathbb{P}^n}(d)$ to V. By the Whitney sum formula for the total Chern class, we have

$$c(\Theta_V) \cdot c(N_{V/\mathbb{P}^n}) = c(\Theta_{\mathbb{P}^n}|_V).$$

Denote by ι the inclusion map $V \to \mathbb{P}^n$. By the naturality of Chern class, $c(\Theta_{\mathbb{P}^n}|_V) = \iota^* c(\Theta_{\mathbb{P}^n})$. But $c(\Theta_{\mathbb{P}^n})$ can be computed in terms of the exponential sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+1)} \to \Theta_{\mathbb{P}^n} \to 0$$

(to remember it, simply memorize $\Theta_{\mathbb{P}^n} = \mathcal{H}om(\mathcal{O}(-1), \mathcal{O}^{\oplus(n+1)}/\mathcal{O}(-1))$). Thus

$$c(\Theta_{\mathbb{P}^n}) = (1 + c_1(\mathcal{O}_{\mathbb{P}^n}(1)))^{n+1} \text{ in the cohomology ring of } \mathbb{P}^n.$$

Denote by $\mathcal{O}_V(m)$ the restriction of $\mathcal{O}_{\mathbb{P}^n}$ to V, and by $h \in \mathrm{H}^2(V;\mathbb{Q})$ the pullback of $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, i.e., $h = c_1(\mathcal{O}_V(1))$. Recall the adjunction formula asserts that $\mathcal{O}_V(d) = N_{Z/\mathbb{P}^n}$. Thus

$$c(\Theta_V) = \frac{(1+h)^{n+1}}{1+dh}.$$

Since V has degree d, we have

$$\int_V h^{n-1} = d.$$

Thus $\chi(V)$ equals d times the coefficient of h^{n-1} in the power series expansion of

$$(1+h)^{n+1}(1 - dh + d^2h^2 - \cdots).$$

This number is easy to compute, it is

$$\begin{aligned}
&d\cdot(-1)^{n-1}\left[d^{n-1} - \binom{n+1}{1}d^{n-2} + \cdots + \binom{n+1}{n-1}(-1)^{n-1}\right]\\
&= (-1)^{n-1}\frac{(d-1)^{n+1} + (-1)^n}{d} + (n+1)\\
&= (-1)^{n-1}\frac{(d-1)^{n+1} + (-1)^n}{d} + \chi(\mathbb{P}^n).
\end{aligned}$$

Therefore, we conclude that the dimension of the $(n-1)^{\text{th}}$ *primitive cohomology* of V, i.e., the cokernel of $\mathrm{H}^{n-1}(\mathbb{P}^n) \to \mathrm{H}^{n-1}(V)$, is

$$\boxed{\dim \mathrm{H}^{n-1}_{\text{prim}}(V;\mathbb{Q}) = \frac{(d-1)^{n+1} + (-1)^n}{d}}. \tag{1.3}$$

2. What about singular hypersurfaces?

What can we say about Betti numbers of singular hypersurfaces? If singularities present, we no longer have $\dim \mathrm{H}^{n-1+i}(V;\mathbb{Q}) = \dim \mathrm{H}^{n-1-i}(V)$ due to the failure of Poincaré duality. Without delving into the study of possible singularities, can we get a good bound of these Betti numbers for *any* degree d hypersurface?

There *is* such a universal bound, i.e.,

Proposition 2.1. *There exists a constant $C(n,d,i)$ such that*

$$\dim \mathrm{H}^i(V;\mathbb{Q}) \le C(n,d,i)$$

for any degree d hypersurface in $\mathbb{P}^n$.

Proof. Let $\mathcal{M}_{n,d}$ be the projective space of all degree d homogeneous polynomials in $x_0,\ldots,x_n$. A point $[F]$ of $\mathcal{M}_{n,d}$ corresponds to a unique hypersurface $\{F=0\}$ in $\mathbb{P}^n$. Let

$$\mathcal{V}_{n,d} = \{(x,[F]) \in \mathbb{P}^n \times \mathcal{M}_{n,d} : F(x) = 0\}.$$

Then $\mathcal{V}_{n,d}$ is a (nonsingular) projective variety, and admits a (necessarily projective) morphism $\pi\colon \mathcal{V}_{n,d} \to \mathcal{M}_{n,d}$ (restriction of the the projection to the second factor). By proper base change, the stalk of $R^i\pi_*\mathbb{Q}$ at $[F]$ equals H^i of the hypersurface $\{F=0\}$. Since $R^i\pi_*\mathbb{Q}$

is constructible (a nontrivial theorem!), there is a stratification of $\mathcal{M}_{n,d}$ by locally closed subvarieties indexed by a finite set A,

$$\mathcal{M}_{n,d} = \bigsqcup_{\alpha \in A} U_\alpha$$

such that $R^i\pi_*\mathbb{Q}|_{U_\alpha}$ is locally constant, hence of constant rank, say b^i_α. We can then take

$$C(n,d,i) = \max\{b^i_\alpha : \alpha \in A\}.$$ □

The problem is how large $C(n,d,i)$ can be. The following theorem is due to Maxim, Păunescu, and Tibăr 2022.

Theorem 2.2. *Let notation be as above. let V be any smooth degree d hypersurface in $\mathbb{P}^n$. Then we have*

$$C(n,d,n-1) \leq \dim \mathrm{H}^{n-1}(V;\mathbb{Q}) \leq \frac{(d-1)^{n+1} + (-1)^n}{d} + 1$$

Note that H^{n-1} is the "crucial" middle degree of the cohomology, and in principle it is the hardest to deal with. Once we get a control of $C(n,d,n-1)$, we can get a control over $C(n,d,n-1+i)$ for all $i \geq 0$, via some sort of generic Lefschetz argument.

I want to prove this theorem in this lecture and also explain how to use it to get other Betti bounds. The purpose is really to demonstrate of the power of the formalism of vanishing cycles and its interaction with perverse sheaves. We will not really need the homological algebra of perverse sheaves. We only need to know that vanishing cycle preserves perversity (which implies that, on a complete intersection X of pure dimension m, $\mathbb{Q}_X[m]$ is perverse), and the Artin vanishing theorem for perverse sheaves. In fact, in most applications, these are all we need.

Let me sketch the idea of proof.

To begin with, recall that for a regular function $f\colon X \to \mathbb{A}^1$, we have a *canoncial triangle*

$$\mathbb{Q}_{X_0} \to R\Psi_f\mathbb{Q}_X \to R\Phi_f\mathbb{Q}_X,$$

where $X_t = f^{-1}(t)$. If f is proper, then $R\Gamma(X_0; R\Psi_f\mathbb{Q})$ computes the cohomology $R\Gamma(X_t;\mathbb{Q})$ of a "fiber sufficiently close to X_0". Thus, by taking cohomology, we see the difference between $\mathrm{H}^*(X_0)$ and $\mathrm{H}^*(X_t)$ is measured by $\mathrm{H}^{*-1}(X_0; R\Phi_f\mathbb{Q})$ and $\mathrm{H}^*(X_0; R\Phi_f\mathbb{Q})$.

Now if X is a variety on which $\mathbb{Q}_X[n]$ is perverse, then $R\Phi_f\mathbb{Q}[n-1]$ is a perverse sheaf on X_0. We also know $R\Phi_f\mathbb{Q}$ only supports on the critical locus, so there is a possibility that the $R\Phi_f\mathbb{Q}_x$ is non-zero only for x contained in an affine subvariety U of X_0. If so, Artin vanishing will be applicable!

Back to the hypersurface case. If $V_0 = \{F_0 = 0\}$ is a fixed singular hypersurface, the idea is to compare the cohomology of V_0 with a the cohomology of a *generic* nonsingular hypersurface $V_\infty = \{F_\infty = 0\}$. Consider the pencil

$$\mathcal{V} = \{(x,t) \in \mathbb{P}^n \times \mathbb{A}^1 : F_0(x) + tF_\infty(x) = 0\}.$$

Then the projection to $\mathbb{A}^1$ induces a proper morphism

$$f\colon \mathcal{V} \to \mathbb{A}^1.$$

Since $\mathcal{V}$ is a hypersurface in a smooth variety $\mathbb{P}^n \times \mathbb{A}^1$, we conclude that $\mathbb{Q}_{\mathcal{V}}[n]$ is a perverse sheaf. This implies that $R\Phi_f(\mathbb{Q}_{\mathcal{V}})[n-1]$ is a perverse sheaf on V_0.

On the other hand, the *base locus* $B = V_0 \cap V_\infty$ appears in $V_t = f^{-1}(t)$ for any t. We have the following theorem (Maxim, Saito, and Schürmann 2013):

Theorem 2.3. *If V_∞ is generic, then we have $R\Phi_f \mathbb{Q}_{\mathcal{V}}|_B = 0$.*

Thus, $R\Phi_f(\mathbb{Q}[n-1]) = j_! \mathcal{P}$ for some complex $\mathcal{P}$ on $V_0 \setminus B$, an affine variety. But we have the following simple

Lemma 2.4. *Let $j\colon U \to X$ be an open immersion of varieties. If $j_!\mathcal{F}$ satisfies the support condition, so is $\mathcal{F}$. If $Rj_*\mathcal{F}$ satisfies the support condition, so is $\mathcal{F}$.*

Thus $\mathcal{P}$ is itself must be a perverse sheaf. By Artin vanishing,

$$\mathrm{H}^i(R\Phi_f(\mathbb{Q})[n-1]) = \mathrm{H}^i_c(V_0 \setminus B; \mathcal{P}) = 0$$

for all $i < 0$. Hence $\mathrm{H}^n(V_0) \to \mathrm{H}^n(V_0; R\Psi_f\mathbb{Q}) \cong \mathrm{H}^n(V_t)$ is injective.

Arithmetic motivation. Let me briefly sketch a motivation for doing this sort of business, beyond mere curiosity about topology of algebraic varieties. If V is a degree d hypersurface in $\mathbb{P}^n$ over an algebraic closure k of a finite field k, and the defining equation of V falls in $\mathbb{F}_q[x_0, \ldots, x_n]$. Then a task in number theory is to understand the behavior of the number of solutions of $F = 0$ in all $\mathbb{F}_{q^m}$. The *zeta function* of V is defined as

$$Z_V(t) = \exp\left\{\sum_{m=1}^{\infty} \frac{\operatorname{Card} V(\mathbb{F}_{q^m})}{m} t^m\right\}.$$

It is known (Dwork 1960) that $Z_V(t)$ is rational fraction:

$$Z_V(t) = \frac{(1-\alpha_1 t)\cdots(1-\alpha_{d_1} t)}{(1-\beta_1 t)\cdots(1-\beta_{d_2} t)}.$$

The numbers α_i and β_j are in fact algebraic integers, and are called the characteristic roots of $Z_V(t)$. Plainly:

$$\operatorname{Card} V(\mathbb{F}_{q^m}) = \beta_1^m + \cdots + \beta_{d_2}^m - \alpha_1^m - \cdots - \alpha_{d_1}^m.$$

So to get hold on the cardinality, I need to know how large the characteristic roots are. Deligne 1980 gives a quite good of understanding of the arithmetic nature of $|\alpha_i|$ and $|\beta_j|$ ("theory of weights"); it turns out that these are the so-called Weil q-numbers, i.e., $|\alpha_i| = q^{a_i/2}$ for some a_i, and $|\beta_j| = q^{b_j/2}$ for some b_j, and the same also hold for all of their complex conjugates. Deligne's result, when understood properly, gives the *order* of the "error" term of $|\operatorname{Card} V(\mathbb{F}_q) - \operatorname{Card} \mathbb{P}^{n-1}(\mathbb{F}_q)|$.

To get more precise information of the difference, we also need to know the magnitude of the major error term. The rough version is this: miraculously, $d_1 + d_2$ is no larger than the total Betti numbers of V (calculated using the ℓ-adic cohomology). An appendix of Hartshorne 1977 contains a more detailed introduction to these ideas.

3. Vanishing cycles along base locus

Let Y be a smooth projective variety. Let X_0 be a strict normal crossing divisor on Y. Let X_∞ be a divisor on Y linearly equivalent to X_0, such that X_∞ intersects any intersection of irreducible components of X_0 transversely. Let $\{X_t\}$ be the pencil spanned by X_0 and X_∞. Let $Z = X_0 \cap X_\infty$ be the base locus of the pencil. Form the incidence variety

$$\mathcal{X} = \{(x,t) \in Y \times \mathbb{P}^1 : x \in X_t\}.$$

Let $\pi\colon \mathcal{X} \to \mathbb{P}^1$ be the natural projection. Let t be a coordinate around $0 \in \mathbb{P}^1$, and $f = t \circ \pi$.

Lemma 3.1. *In the situation above, we have*

$$\boxed{R\Phi_f(\mathbb{Q}_\mathcal{X})|_Z = 0}.$$

Proof. Let z be any point of Z. Let us prove that the stalk of $R\Phi_f(\mathbb{Q}_\mathcal{X})$ at z vanishes. The condition that X_∞ intersects all the intersections of X_0 transversely is equivalent to saying that $X_0 + X_\infty$ is a strict normal crossing divisor. Choose a coordinate system $(y_1, \ldots, y_n)$ of Y centered at z so that locally around z the equation of X_0 is $\prod_{i=1}^r y_i^{m_i}$, and X_∞ being cut out by $y_n = 0$. The equation of $\mathcal{X}$, in a neighborhood of 0 in $\mathbb{A}^n \times \mathbb{A}^1_t$ is then $\prod_{i=1}^r y_i^{m_i} + t y_n = 0$. For $0 < |t| \ll \epsilon \ll 1$, the Milnor fiber may be identified with

$$\left\{(y_1, \ldots, y_n) \in \mathbb{C}^n : \sum_{i=1}^n |y_i|^2 < \epsilon^2, t y_n = \prod_{i=1}^r y_i^{m_i}\right\}$$

which is homeomorphic to

$$F = \left\{(y_1, \ldots, y_{n-1}) \in \mathbb{C}^{n-1} : \sum_{i=1}^{n-1} |y_i|^2 + |t|^{-2} \prod_{i=1}^r |y_i|^{2m_i} < \epsilon^2\right\}.$$

We can define a deformation retract of F to a single point by

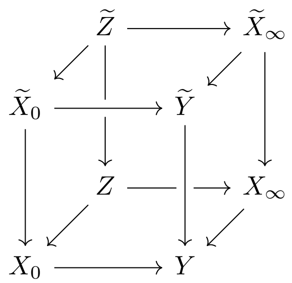
$$F \times [0,1] \to F, \quad (y_1, \ldots, y_{n-1}) \mapsto (\lambda y_1, \ldots, \lambda y_{n-1}).$$

This shows that $R\Phi_f(\mathbb{Q}_\mathcal{X})_z = 0$. □

Theorem 3.2. *Let Y be a projective variety, possibly singular. Let X_0 be a* very ample *divisor on Y. We assume that the complement $Y \setminus X_0$ is a* smooth *affine variety. Let $\{X_t\}_{t \in \mathbb{P}^1}$ be a pencil of divisors linearly equivalent to X_0. Assume that X_∞ is chosen sufficiently generically. Let $\mathcal{X} = \{(x,t) \in Y \times \mathbb{P}^1 : x \in X_t\}$, with a natural projection $\pi\colon \mathcal{X} \to \mathbb{P}^1$. Let Z be the base locus of this pencil. Let t be a coordinate of $\mathbb{A}^1$, and $f = t \circ \pi$. Then*

$$\boxed{R\Phi_f(\mathbb{Q}_\mathcal{X})|_Z = 0}.$$

Proof. Let $\sigma\colon (\widetilde{Y}, \widetilde{X}_0) \to (Y, X_0)$ be an embedded resolution. Thus $\widetilde{X}_0$ is a divisor with strict normal crossing. The generic condition on X_∞ is that we require $\widetilde{X}_\infty = \pi^* X_\infty$ intersects all the irreducible components of $\widetilde{X}_0$ transversely. We can form the pencil $\widetilde{\mathcal{X}} = \bigcup \widetilde{X}_t$. Let $\widetilde{Z}$ be the base locus of this pencil. Then we have a commutative diagram



in which every square is cartesian. Let $U = \mathbb{A}^1 \setminus \{0\}$, $\widetilde{\mathcal{X}}_U = \widetilde{\mathcal{X}} \times_{\mathbb{P}^1} U$, and similarly define $\mathcal{X}_U$. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}^\circ & \xrightarrow{\widetilde{j}_U} & \widetilde{\mathcal{X}}_U & \xleftarrow{\widetilde{i}_U} & \widetilde{Z} \times U \\ \| & & \downarrow{\scriptstyle \sigma_U} & & \downarrow{\scriptstyle \sigma_Z \times \mathrm{Id}} \\ \mathcal{X}^\circ & \xrightarrow{j_U} & \mathcal{X}_U & \xleftarrow{i_U} & Z \times U \\ & & \downarrow{\scriptstyle f} & & \\ & & U & & \end{array}.$$

One last bit of notation: for each $t \in \mathbb{A}^1$, let $\sigma_t \colon \widetilde{X}_t \to X_t$ be the natural map.

The distinguished triangle

$$\widetilde{j}_{U!}\mathbb{Q}_{\widetilde{\mathcal{X}}^\circ} \to \mathbb{Q}_{\widetilde{\mathcal{X}}_U} \to \widetilde{i}_{U*}\mathbb{Q}_{\widetilde{Z}\times U} \to$$

gives a distinguished triangle

$$R\Psi_{f\circ\sigma_U}(\widetilde{j}_{U!}\mathbb{Q}_{\widetilde{\mathcal{X}}^\circ}) \to R\Psi_{f\circ\sigma_U}(\mathbb{Q}_{\widetilde{\mathcal{X}}_U}) \to R\Psi_{f\circ\sigma_U}(\widetilde{i}_{U*}\mathbb{Q}_{\widetilde{Z}\times U}) \to .$$

Since $\widetilde{i}_U$ is proper, $\Psi_{f\circ\sigma_U}(\widetilde{i}_{U*}\mathbb{Q}) = \widetilde{i}_{0*}\Psi_{f\circ(\sigma_Z\times\mathrm{Id})}(\mathbb{Q})$, which is isomorphic to $\widetilde{i}_{0*}\mathbb{Q}_{\widetilde{Z}}$ via the canonical map. It follows Lemma 3.1 that

$$R\Psi_{f\circ\sigma_U}(\widetilde{j}_{U!}\mathbb{Q})|_{\widetilde{Z}} = 0.$$

Therefore,

$$\begin{aligned} R\Psi_f(j_{U!}\mathbb{Q})|_Z &= R\Psi_f(\sigma_{U*}\widetilde{j}_{U!}\mathbb{Q})|_Z \\ &= \sigma_{0*}(R\Psi_{f\circ\sigma_U}(\widetilde{j}_{U!}\mathbb{Q}))|_Z \\ &= \sigma_{0*}(R\Psi_{f\circ\sigma_U}(\widetilde{j}_{U!}\mathbb{Q})|_{\widetilde{Z}}) \\ &= 0. \end{aligned}$$

It follows that the natural map $R\Psi_f(\mathbb{Q}) \to R\Psi_f(i_{U*}\mathbb{Q})$ is an isomorphism. Since the latter is naturally isomorphic to $i_{0*}\mathbb{Q}_Z$, we conclude that the natural map $\mathbb{Q}_Z \to R\Psi_f(\mathbb{Q})|_Z$ is an isomorphism. This implies that $R\Phi_f(\mathbb{Q})|_Z = 0$, as desired. □

4. Cohomology beyond middle dimension

A Gysin Lemma. Now let us consider cohomology of a hypersurface V in $\mathbb{P}^n$ when the cohomological degree i is bigger than $n-1$. The idea is that a Lefschetz type theorem should hold.

We already know that for a perverse sheaf $\mathcal{P}$ on a variety X, we have the following Lefschetz theorem: for *any* hyperplane A, we have

$$\mathrm{H}^i(X;\mathcal{P}) \to \mathrm{H}^i(A\cap X;\mathcal{P})$$

is bijective if $i < -1$, and surjective if $i = -1$. Taking dual implies that for a perverse sheaf $\mathcal{P}$ and *any* hyperplane A, we have

$$\mathrm{H}^i(A\cap X;\iota^!\mathcal{P}) \to \mathrm{H}^i(X;\mathcal{P}).$$

is surjective if $i = 1$ and bijective if $i > 1$

Now if V is a hypersurface in $\mathbb{P}^n$, and $\mathcal{P} = \mathbb{Q}_V[n-1]$, then for any A, $A \cap V = V'$ is a lower dimensional hypersurface. The above says that

$$\mathrm{H}^{n-1+i}(V'; \iota^! \mathbb{Q}_V) \to \mathrm{H}^{n-1+i}(V; \mathbb{Q})$$

is surjective. If somehow we can arrange $\iota^! \mathbb{Q}_V = \mathbb{Q}_{V'}[-2]$, just like the smooth case, then we can use induction and the estimates for the middle cohomology of lower dimensional hypersurfaces to get upper bounds for $\dim \mathrm{H}^{n-1+i}(V)$.

All I want above are OK. First, there is indeed a natural "purity map"

$$\iota^* \mathbb{Q}_V[-2] \to \iota^! \mathbb{Q}_V,$$

which I will construct below. With this map canonically defined, we have the following generic purity theorem of Deligne.

Theorem 4.1 (Generic purity)**.** *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety. Let $\mathcal{F}$ be a constructible complex on X. Let $\iota\colon A \to X$ be a hyperplane section. Then the purity map $\iota^* \mathcal{F}[-2](-1) \to \iota^! \mathcal{F}$ is an isomorphism.*

Construction of the purity map. Let $\mathbb{P}$ be a nonsingular variety of dimension N. Let H be a smooth subvariety of codimension r. Then we shall first construct, for any constructible complex $\mathcal{F}$ on $\mathcal{P}$, a *purity map*

$$\kappa_{\mathcal{F}}\colon \iota_H^* \mathcal{F} \to \iota_H^! \mathcal{F}[2r](r). \tag{4.2}$$

For this, by adjunction, it suffices to construct a morphism

$$\begin{array}{ccc} \iota_{H*}\iota_H^* \mathcal{F} & \xrightarrow{\mu_{\mathcal{F}}} & \mathcal{F}[2r](r) \\ \| & & \| \\ \iota_{H*}\iota_H^* \mathbb{Q} \otimes \mathcal{F} & \underset{\mu_{\mathbb{Q}} \otimes \mathrm{Id}}{\dashrightarrow} & \mathbb{Q}[2r](r) \otimes \mathcal{F} \end{array} \tag{4.3}$$

If we know how to define a morphism $\kappa_{\mathbb{Q}}$, hence a morphism $\mu_{\mathbb{Q}}$, we simply define $\mu_{\mathcal{F}} = \mu_{\mathbb{Q}} \otimes \mathrm{Id}_{\mathcal{F}}$ by means of the canonical vertical isomorphisms in (4.3). For this, have the following relative purity theorem.

Proposition 4.4 (Purity theorem)**.** *In the situation above, if $\mathcal{F}$ is lisse, then there is a Gysin isomorphism $\kappa\colon \iota_H^! \mathbb{Q}_{\mathbb{P}} \to \mathbb{Q}_H[2r](r)$*

Proof. Exercise. □

Now let us consider the situation when $X \subset \mathbb{P}^N$ is quasi-projective, and A is a hyperplane section. In this case, let $\mathbb{P}$ be an open subscheme of $\mathbb{P}^N$ such that X is a closed subscheme of $\mathbb{P}$. Let $u\colon X \to \mathbb{P}$ be the inclusion map. We form the following cartesian diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ H & \xrightarrow{\iota_H} & \mathbb{P}. \end{array}$$

By our previous discussion, there is a purity map

$$\begin{array}{ccc} \iota_H^* u_* & \xrightarrow{\kappa} & \iota_H^! u_*[2](1) \\ \| & & \| \\ v_* \iota^! & & v_* \iota^![2](1). \end{array}$$

Since v is a closed embedding, applying v^* to the above gives the desired purity map $\iota^* \to \iota^![2](1)$.

Generic purity theorem. In general (without lissity of $\mathcal{F}$ or the smoothness of X and D), the purity theorem, Proposition 4.4 fails to hold. For example, we can take D to be a hypersurface of $X = \mathbb{P}^n$, which fails the Poincaré duality and $\mathcal{F} = \mathbb{Q}_X$. However, if the divisor is chosen in some sense "generically", then the purity map can still be rendered to be an isomorphism. This is known as the *generic purity theorem.* This theorem is closely related to Deligne's generic base change theorem (SGA $4\frac{1}{2}$, Finitude). Let us thus start by proving some generic base change lemmas.

Lemma 4.5. *Let X be a nonsingular algebraic variety. Let $j\colon U \to X$ be an open immersion, such that $D = X \setminus U$ is a divisor with strict normal crossings.*[1] *Let A be a Cartier divisor of X such that A is transverse to $\bigcap_{\alpha\in J} D_\alpha$ for any nonempty subset J of I. Form the fiber diagram*

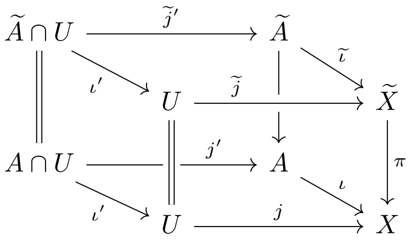
$$\begin{array}{ccc} A \cap U & \xrightarrow{j'} & A \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\ U & \xrightarrow{j} & X \end{array} .$$

Then the base change map $\iota^* Rj_*\mathcal{E} \to Rj'_*\iota'\mathcal{E}$ is an isomorphism.

Proof. Exercise. □

Lemma 4.6. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety. Let $j\colon U \to X$ be an open immersion. Assume that U is nonsingular, and let $\mathcal{E}$ be a local system on U. Then for a generic hyperplane section $\iota\colon A \to X$, the base change map $\iota^* Rj_*\mathcal{E} \to Rj'_*\iota'\mathcal{E}$ is an isomorphism.*

Proof. Let $\pi\colon \widetilde{X} \to X$ be a resolution of singularity such that $\pi^{-1}U \to U$ is an isomorphism, and such that $\pi^{-1}(X \setminus U) = \bigcup_{\alpha\in I} D_\alpha$ is a divisor with strict normal crossings. We choose a hyperplane A such that the total transform $\widetilde{A}$ of A is transverse to all $\bigcap_{\alpha\in J} D_\alpha$, where J ranges through all nonempty subsets of I. Such an A exists thanks to the base-point-free Bertini theorem. We then get the following commutative diagram



[1]That is, $D = \bigcup_{\alpha\in I} D_\alpha$ is a union of irreducible nonsingular varieties; for any nonempty subset J of I, the intersection $D^{(J)} = \bigcap_{\alpha\in J} D_\alpha$ is nonsingular; and for any $x \in D^{(J)}$, $\dim_x D^{(J)} = \dim_x X - \operatorname{Card} J$.

in which all squares are cartesian. By Lemma 4.5, we have

$$\widetilde{\iota}^* R\widetilde{j}_*\mathcal{E} \simeq R\widetilde{j}'_*\iota'^*\mathcal{E}, \tag{4.7}$$

It follows that

$$\begin{aligned} \iota^* Rj_*\mathcal{E} &= \iota^* R\pi_* R\widetilde{j}_*\mathcal{E} \\ &= R\pi|_{\widetilde{A}*}\widetilde{\iota}^* R\widetilde{j}_*\mathcal{E} && \text{(proper base change)} \\ &= R\pi_{\widetilde{A}*} R\widetilde{j}'_*\iota'^*\mathcal{E} && (4.7) \\ &= Rj'_*\iota'\mathcal{E}. \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 4.1. Let us first prove the theorem for $\mathcal{F} = Rj_*\mathcal{E}$ in the situation above, where $j\colon U \to X$ is the open immersion of a nonsingular open subset of X. We have

$$\begin{aligned} \iota^* Rj_*\mathcal{E} &\simeq Rj'_*\iota'^*\mathcal{E} && \text{by Lemma 4.6} \\ &\simeq Rj'_*\iota'^!\mathcal{E}[2](1) && \text{by Proposition 4.4} \\ &\simeq \iota^! Rj_*\mathcal{E}[2](1) && \text{by proper base change.} \end{aligned}$$

For the general case we can use noetherian induction. If $\mathcal{F}$ is a constructible complex, we choose a smooth open subset $j\colon U \to X$ such that $\mathcal{F}|_U$ is lisse. Let $i\colon Z \to X$ be the complement of U. Then $i_*i^!\mathcal{F}$ is constructible, and is supported on a proper closed subset of X. The inductive hypothesis implies that Theorem 4.1 holds for $i_*i^!\mathcal{F}$, and the argument above in the preceding paragraph implies that Theorem 4.1 holds for $Rj_*j^*\mathcal{F}$. The theorem then follows from the five lemma. □

Remark 4.8. I do not know how to transplant the proof above to positive characteristics. Although one can use alteration in place of the resolution of singularity, the base-point-free Bertini theorem is not true over a field of positive characteristic. Nevertheless, Theorem 4.1 can be deduced from Deligne's generic base change theorem.

Now we can produce some examples of perverse sheaves from a given one.

Proposition 4.9. *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. Let $\mathcal{P}$ be a perverse sheaf on X. Then for a generic hyperplane section $\iota\colon A \to X$, $\iota^*\mathcal{P}[-1]$ is a perverse sheaf.*

Proof. Let $X = \bigsqcup Z_\alpha$ be a stratification for $\mathcal{P}$. Then the support condition of $\mathcal{P}$ is equivalent to $\mathcal{H}^m\iota^*_{Z_\alpha}\mathcal{P} \le -\dim Z_\alpha$. If A is generic, then $Z_\alpha \cap A$ can be arranged to be smooth, with dimension one lower than $\dim Z_\alpha$, and $A = \bigsqcup(Z_\alpha \cap A)$ is a stratification for $\iota^*\mathcal{P}[-1]$. Thus the support condition of $\iota^*\mathcal{P}[-1]$ is verified.

The cosupport condition of $\iota^*\mathcal{P}[-1]$ is the same as the support condition of $\iota^!(\mathbb{D}\mathcal{P})[1]$. If A is generic, the generic purity implies that $\iota^!(\mathbb{D}\mathcal{P})[1] \cong \iota^*(\mathbb{D}\mathcal{P})[-1](-1)$. Applying the previous paragraph to $\mathbb{D}\mathcal{P}$ then finishes the proof. □

5. Algebraic proof of the key lemma

Let R be an equal-characteristics strictly henselian discrete valuation ring with residue field k and field of fractions K. We fix a separable closure K^{sep} of K and a uniformizer μ of

R. Let $S = \operatorname{Spec} R$ be the corresponding trait, with special point $s = \operatorname{Spec} k$, generic point $\eta = \operatorname{Spec} K$, and geometric generic point $\overline{\eta} = \operatorname{Spec} K^{\mathrm{sep}}$.

Let X be a smooth scheme over k, and f, g are two regular functions on X. Then we can form the pencil spanned by f and g

$$\widetilde{X} = \{(x, t) \in \mathbb{A}^1 : f(x) - tg(x) = 0\}$$

which admits a morphism $t\colon \widetilde{X} \to \mathbb{A}^1$. Let 0 be a point of X such that $f(0) = g(0) = 0$. Assume that there is a étale coordinate system around 0, with respect to which f and g are monomials, i.e., there exists an open neighborhood U of 0, an étale morphism $c\colon U \to \mathbb{A}^n_k \times \mathbb{A}^m_k$ (the first $\mathbb{A}^n$ is coordinated by $x_1, \ldots, x_n$, the second $\mathbb{A}^m$ is coordinated by $y_1, \ldots, y_m$), such that $f|_U = (x \mapsto x_1^{b_1} \cdots x_r^{b_r}) \circ \mathrm{pr}_1 \circ c$, $g|_U = (y \mapsto y_1^{a_1} \cdots y_s^{a_s}) \circ \mathrm{pr}_2 \circ c$.

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