PROBLEM SET 1

1. Let $M = \mathbb{R}^2$. Let $X_1 = \{(x, y) \in M : y = 0\}$, $X_2 = \{(x, y) \in M : xy = 0\}$, and let $i_a : X_a \to M$ be the inclusion. Define $\mathcal{F}_a = i_{a*}\mathbb{Q}_{X_a}$, $a = 1, 2$, and define $f_1(x, y) = y - x^2$, $f_2(x, y) = y$, $f_3 = 3y - 3x$. For any $a = 1, 2; b = 1, 2, 3$, determine the stalk $i_b^!(\mathcal{F}_a)_{(0,0)}$, where i_b is the inclusion of $\{f_b \geq 0\}$ in M.

2. Let M be an *orientable* manifold. Let $i : N \to M$ be a closed submanifold of M of codimension r.

(i) Show that $\mathcal{H}^r i^! \mathbb{Q}_M$ is a local system (locally constant sheaf) on N, and $\mathcal{H}^m i^! \mathbb{Q}_M = 0$ for all $m \neq r$. Give an example that $\mathcal{H}^r i^! \mathbb{Q}_M$ is not constant. [*Hint.* The sheaf $\mathcal{H}^r i^! \mathbb{Q}_M$ is constant if and only if N is orientable.]

(ii) Deduce that if $\mathcal{H}^r i^! \mathbb{Q}_N$ is constant, then for any local system $\mathcal{F}$ on M, $\mathrm{H}^m_N(M; \mathcal{F}) = \mathrm{H}^{m-r}(N; \mathcal{F}|_N)$.

(iii) Look up the *Thom isomorphism theorem.* What is the relation between this theorem and (ii)?

3. Review Mittag-Leffler condition as presented in, e.g., Weibel, §3.5. Prove the following stronger version of the Vietoris–Begle theorem. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that X is a union $X = \bigcup X_n$, such that

- for any n, $f|_{X_n}$ is proper, and its fibers are acyclic,
- X_n is contained in the interior of X_{n+1}.

Then for any sheaf $\mathcal{F}$ on Y, we have $\mathcal{F} \simeq Rf_* f^* \mathcal{F}$. (See Kashiwara and Schapira, Proposition 2.7.8.)

References

Kashiwara, Masaki and Pierre Schapira (1990), *Sheaves on manifolds*, vol. 292, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], With a chapter in French by Christian Houzel, Springer-Verlag, Berlin, pp. x+512.

Weibel, Charles A. (1994), *An introduction to homological algebra*, vol. 38, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, pp. xiv+450.