PROBLEM SET 2

1. This problem asks you to compute explicitly the dimension of certain cohomology groups.

(i) Let Σ be the suspension of $S^1 \times S^1$. Calculate $\dim \mathrm{H}^*(\Sigma;\mathbb{Q})$. This is a simple example of a singular space on which Poincaré duality fails.

(ii) Calculate the compactly supported cohomology of the half open interval $]a,b]$.

(iii) Let C be the open cone over a 2-dimensional torus, i.e., the quotient space

$$C = \frac{(S^1 \times S^1) \times [0,1[}{(S^1 \times S^1) \times \{0\}}.$$

Show that the Poincaré duality (between cohomology and compactly supported cohomology) fails for C.

(iv) Let $X = \{[x,y,z,w] \in \mathbb{P}^3 : x^4 + y^4 + z^4 = 0\}$. Show that X is singular at $[0,0,0,1]$. Calculate $\mathrm{H}^*(X;\mathbb{Q})$.

2. Jouanolou's device. Let X be a quasi-projective variety. Construct an affine variety E, and a smooth sumorphism $\pi\colon E \to X$, such that the fibers of π are all affine spaces. Explain why such a $\pi\colon E \to X$ cannot be a vector bundle over X in general. Conclude that any quasi-projective variety is homotopy equivalent to an affine variety.

[*Hint.* First prove it for $\mathbb{P}^n$. In this case, E can be taken as the complement of the universal hyperplane of $\mathbb{P}^n \times \mathbb{P}^n$. With a further argument, show that it suffices to deal with quasi-affine X. If X is the complement of $f_1 = \cdots = f_r = 0$ inside an affine variety Z, show that we can take $E \subset Z \times \mathbb{A}^r_{y_1,\ldots,y_r}$ as defined by $\sum y_i f_i = 0$.]

3. Recall the theory of spectral sequence for filtered complexes (consult, for example, Weibel 1994, §5.4 or Voisin 2002, §8.3). Let Y be a topological space. Let $Y_0 \subset Y_1 \subset \cdots \subset Y_n = Y$ be a filtration of Y by closed subspaces. Let $\mathcal{F}$ be a sheaf on Y. Construct a spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q}(Y_p, Y_{p-1}; \mathcal{F}|_{Y_p}) \Rightarrow \mathrm{H}^{p+q}(Y;\mathcal{F}).$$

Deduce that for an n-dimensional CW complex Y, with Y_p its p-dimensional skeleton, $\mathcal{F}$ the constant sheaf $\mathbb{Z}_Y$, the above spectral sequence reduces to the usual complex computing the cellular cohomology.

4. Let $f\colon Y \to X$ be a smooth projective morphism of algebraic varieties. Assume that X is affine, $X_\bullet$ a cellular decomposition for the sheaves $R^i f_* \mathbb{Q}_Y$. Let $Y_i = f^{-1} X_i$ and $\mathcal{F} = \mathbb{Q}_Y$. In this concrete situation, write down what $E_1^{p,q}$ and $E_2^{p,q}$ are.

References

Voisin, Claire (2002), *Théorie de Hodge et géométrie algébrique complexe*, vol. 10, Cours Spécialisés [Specialized Courses], Société Mathématique de France, Paris, pp. viii+595.

Weibel, Charles A. (1994), *An introduction to homological algebra*, vol. 38, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, pp. xiv+450.