PROBLEM SET 3

The purpose of this problem set is to prove the following proposition that was used in the class.

Proposition. *Let S be a finite subset of $\mathbb{A}^1$. Let $j\colon \mathbb{A}^1 \setminus S \to \mathbb{A}^1$ be the inclusion map. Then for any local system $\mathcal{E}$ on $\mathbb{A}^1 \setminus S$, we have $\mathrm{H}^m(\mathbb{A}^1; j_!\mathcal{E}) = 0$ for any $m \neq 1$.*

(1) Suppose we have a commutative diagram

$$\begin{array}{ccc} & & X \\ & \overset{j_X}{\nearrow} & \\ U & & \big\downarrow f \\ & \underset{j_Y}{\searrow} & \\ & & Y \end{array}$$

in which f is proper, j_X and j_Y are open embeddings. Show that

$$\mathrm{H}^m(X; j_{X!}\mathcal{F}) \cong \mathrm{H}^m(Y; j_{Y!}\mathcal{F})$$

for any m and any sheaf $\mathcal{F}$.

(2) Let $r < R$ be two positive real numbers. Consider the following inclusions:
- $\alpha\colon A := \{|z| = R\} \to B := \{r \leq |z| \leq R\} \subset \mathbb{A}^1$,
- $\beta\colon A \setminus \{R\} \to B$.

Calculate $\mathcal{H}^m \alpha^! \beta_! \mathbb{Q}$.

(3) Suppose $\mathcal{E}$ is a locally constant sheaf on $\mathbb{A}^1 \setminus \{0\}$ of rank one.[1] Let $j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ be the open immersion. Calculate $\mathrm{H}^m(\mathbb{A}^1 \setminus \{0\}; \mathcal{E})$, $\mathrm{H}^m(\mathbb{A}^1; j_!\mathcal{E})$, and $\mathrm{H}^m_c(\mathbb{A}^1 \setminus \{0\}; \mathcal{E})$.

(4) Show that any local system on $\mathbb{A}^1 \setminus \{0\}$ is an iterated extension of rank one local systems. Conclude that for any local system $\mathcal{E}$ on $\mathbb{A}^1 \setminus \{0\}$, we have $\mathrm{H}^m(\mathbb{A}^1; j_!\mathcal{E}) = 0$. Here $j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is the inclusion map. This proves the proposition when $\operatorname{Card} S = 1$.

(5) Let $\mathcal{F}$ be a constructible sheaf on $\mathbb{A}^1$. Let $h_R\colon U_R := \{|z| > R\} \to \mathbb{A}^1$ be the inclusion map. Show that if R is sufficiently large, then $\mathrm{H}^m(\mathbb{A}^1; h_{R!}\mathcal{F}|_{U_R}) = 0$ for $i \neq 1$.

(6) Hypotheses as in the proposition, assume S is contained in the closed ball $B_R = \{|z| \leq R\}$. Without loss of generality, we assume that $S \cap \{|z| = R\} = \{s\}$ is a singleton. Use induction on $\operatorname{Card} S$ to prove the proposition. [*Hint.* Use the fundamental distinguished triangle for $\{|z| \leq R\} \to \mathbb{A}^1 \leftarrow \{|z| > R\}$. Note that $\{|z| < R\}$ is diffeomorphic to $\mathbb{A}^1$.]

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[1]these are precisely local systems associated to the multi-valued functions z^α, $\alpha \in [0, 1[$; if you don't know what they are, let me know.