PROBLEM SET 4

1. Let X be an algebraic variety. Recall that a !-stratification of $\mathcal{F} \in D_c^b(X)$ is a stratification $X = \bigsqcup_{\alpha \in A} Z_\alpha$, ($Z_\alpha$ are called *strata* of the stratification) with the following properties:

- $\operatorname{Card} A < \infty$,
- Z_α is a locally closed subset of X with respect to the Zariski topology,
- Z_α is nonsingular and is equidimensional,
- the closure of Z_α is a union of strata.
- if $\iota_\alpha \colon Z_\alpha \to X$ is the inclusion map, then $\mathcal{H}^m \iota_\alpha^! \mathcal{F}$ is a locally constant sheaf on Z_α.

Suppose $\mathcal{F} \in D_c^b(X)$, and $\bigsqcup_{\alpha \in A} Z_\alpha$ is a !-stratification. Show that $\mathcal{F} \in {}^{\mathrm{p}}\mathcal{D}_X^{\geq 0}$ if and only if $\mathcal{H}^m \iota_\alpha^! \mathcal{F} = 0$ for all $m < -\dim Z_\alpha$.

2. Let X be a nonsingular variety, $\pi \colon X \to Y$ a proper morphism. We say π is *semismall* if $\dim X \times_Y X \leq \dim X$ (Recall the dimension of a variety is defined as the largest dimension of its irreducible components). Show that if π semismall, then $R\pi_* \mathbb{Q}_X$ is a perverse sheaf on Y.

3. Consider the inclusion maps $j \colon \mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1$ and $i \colon \{0\} \to \mathbb{A}^1$. Show that all items in the (shifted) fundamental distinguished triangle

$$\mathbb{Q}_{\mathbb{A}^1}[1] \to Rj_* j^* \mathbb{Q}_{\mathbb{A}^1}[1] \to i_* i^! \mathbb{Q}_{\mathbb{A}^1}[1],$$

are perverse sheaves. Is it true that $Rj_* \mathbb{Q}_{\mathbb{G}_{\mathrm{m}}}[1] \approx \mathbb{Q}_{\mathbb{A}^1}[1] \oplus i_* i^! \mathbb{Q}_{\mathbb{A}^1}[1]$?

4. Let S be a normal surface with only one singular point $0 \in S$. Let $j \colon U := S \setminus \{0\} \to S$ be the inclusion map of U. Let $\mathcal{F} = Rj_*(\mathbb{Q}_U[2])$. Show that in general $\mathcal{F}$ is *not* a perverse sheaf. What about $j_!(\mathbb{Q}_U[2])$?