Let $\mathfrak{sl}_2(\mathbb{C})$ be the space of all traceless 2×2 matrices $\left[\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right]$. Form the incidence correspondence

$$\widetilde{X} = \{(A, \ell) \in \mathfrak{sl}_2(\mathbb{C}) \times \mathbb{P}^1 : \ell \text{ is an eigenspace of } A\}.$$

Let $\pi\colon \widetilde{X} \to \mathfrak{sl}_2(\mathbb{C})$ be the restriction of the projection to the first factor. Let U be the subset of $\mathfrak{sl}_2(\mathbb{C})$ consisting of all diagonalizable matrices with two distinct eigenvalues. Let $S = \mathfrak{sl}_2(\mathbb{C}) \setminus U$.

(1) Show that S is isomorphic to the affine cone of a plane conic. Show that the intersection complex of S is isomorphic to $\mathbb{Q}_S[2]$.
(2) Show that $\widetilde{X}$ is a nonsingular 3-dimensional variety, and π is a proper, generically two-to-one morphism.
(3) Show that the restriction of π to $\pi^{-1}(U)$ is a finite étale Galois cover with Galois group $\mathbb{Z}/2$. Conclude that $\pi_*\mathbb{Q}_{\widetilde{X}}$ is a direct sum $\mathcal{L} \oplus \mathbb{Q}_U$, where $\mathcal{L}$ is a nontrivial local system of rank one.
(4) Find the supports of the sheaves $R^i\pi_*\mathbb{Q}_{\widetilde{X}}$. Use your result and relative duality to conclude that $\pi_*\mathbb{Q}_{\widetilde{X}}[3]$ is a perverse sheaf on $\mathfrak{sl}_2(\mathbb{C})$.
(5) Let $V_1 = \mathfrak{sl}_2(\mathbb{C}) \setminus \{0\}$. Let $j_1 : U \to V_1$ be the inclusion. Show that $\tau^{\leq -3}Rj_{1*}(\mathbb{Q}[3])$ is a perverse sheaf $S \setminus \{0\}$. Show also that $\tau^{\leq -3}Rj_{1*}(\mathcal{L}[3]) = Rj_{1*}(\mathcal{L}[3]) = j_{1!}\mathcal{L}[3]$ is a perverse sheaf.
(6) Let $j_2\colon V_1 \to \mathfrak{sl}_2(\mathbb{C})$ be the inclusion map. Show that $\mathcal{I}_1 = \tau^{\leq -2}Rj_{2*}\tau^{\leq -3}Rj_{1*}(\mathbb{Q}[3])$ and $\mathcal{I}_2 = \tau^{\leq -2}Rj_{2*}\tau^{\leq -3}Rj_{1*}(\mathcal{L}[3])$ are perverse sheaves on $\mathfrak{sl}_2(\mathbb{C})$. In fact, show that $\mathcal{I}_1 = \mathbb{Q}_{\mathfrak{sl}_2(\mathbb{C})}[3]$.
(7) Let $X = \widetilde{X} \times_{\mathfrak{sl}_2(\mathbb{C})} S$, and let $\varpi\colon X \to S$ be the second projection. Show that X is nonsingular, and ϖ is a resolution of singularity. Use proper base change and the decomposition $R\pi_*\mathbb{Q}_{\widetilde{X}}[3] = \mathcal{I}_1 \oplus \mathcal{I}_2$ to conclude that $R\varpi_*\mathbb{Q}_X[2] \cong \mathbb{Q}_S[2] \oplus \mathbb{Q}_0$, where $\mathbb{Q}_0$ is the skyscraper sheaf supported at the origin.

It is interesting to notice that the group $\mathbb{Z}/2$ does not act on the X. However, there is a natural action of $\mathbb{Z}/2$ on the perverse sheaves $\mathbb{Q}_S[2]$ and $\mathbb{Q}_0$, corresponding the trivial representation and the sign representation, respectively.