1. Show that the germ $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ has an isolated critical point at 0 if and only if the *Milnor algebra*

$$\frac{\mathbf{C}[\![x_0,\ldots,x_n]\!]}{\left(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right)}$$

is a finite dimensional nonzero $\mathbf{C}$-algebra.

2. Show that the Milnor fiber of the Morse singularity $f(x_0,\ldots,x_n) = x_0^2+\cdots+x_n^2$ is homotopy equivalent to $\mathbf{S}^n$.

3. Let $X = \mathbf{P}^2$, $D = \{x_0x_1x_2 = 0\} \subset \mathbf{P}^2$, and $U = X \setminus D$. Let $f\colon U \to \mathbf{A}^1$ be the function

$$f(x_0,x_1,x_2) = \frac{x_0^3+x_1^3+x_2^3}{x_0x_1x_2}.$$

Find the set $\Sigma \subset U$ of critical points of f. Show these critical points are all locally analytically isomorphic to A_1-singularities.

Let t be any point of $\mathbf{A}^1$ such that $f^{-1}(t)$ has no critical points. Show that $f^{-1}(t)$ is an elliptic curve with 9 points removed. Show that the relative cohomology $\mathrm{H}^i(U, f^{-1}(t);\mathbf{Q})$ is zero if $i \neq 2$, and $\mathrm{H}^2(U, f^{-1}(t);\mathbf{Q}) = \mathrm{Card}\,\Sigma$.

What happens if

$$f(x_0,x_1,x_2) = \frac{x_0^4+x_1^4+x_2^4}{x_0^2x_1x_2}?$$

4. Let $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ be a germ of a holomorphic function with an isolated critical point at 0. Let $f\colon X \to S$ be a representative of f. Let $h\colon Y \to X$ be an embedded resolution, i.e., h is an isomorphism between $Y \setminus h^{-1}(f^{-1}(0))$ and $X \setminus f^{-1}(0)$, $h^{-1}(f^{-1}(0))$ is a divisor with normal crossing: $h^{-1}(f^{-1}(0)) = \sum N_iE_i$ (E_i smooth, irreducible). Set $E_i^\circ = E_i \setminus \bigcup_{j\neq i} E_j$.

Let F be the Milnor fiber of f. Let T be the monodromy operator acting on $\mathrm{H}^n(F;\mathbf{Z})$. Prove A'Campo's formula

$$\det(\lambda I - T|_{\mathrm{H}^n(F)}) = \left((\lambda-1)^{-1}\prod_{i\in I}(\lambda^{N_i}-1)^{\chi(E_i^\circ\cap h^{-1}(0))}\right)^{(-1)^n}.$$

where χ is the Euler characteristic with respect to singular cohomology.