NEARBY CYCLES (1-DIMENSIONAL CASE)

Contents

We have given a provisional definition of the nearby cycle of an integrable connection on Δ^* meromorphic at 0. In this post we give five more interpretations for regular meromorphic connections.

References.

- Deligne, P., and Katz, N. M. *Groupes de monodromie en géométrie algébrique*. II (1973). Exposés XIII and XIV.
- Kashiwara, M. *Vanishing cycle sheaves and holonomic systems of differential equations*. In: Algebraic geometry (Tokyo/Kyoto, 1982) (pp. 134–142).
- Malgrange, B. *Polynômes de Bernstein-Sato et cohomologie évanescente*. In: Analysis and topology on singular spaces, II, III (Luminy, 1981) (pp. 243–267).

1. Nearby cycles as in SGA 7 II

In SGA 7 II, Exposés XIII and XIV, Deligne treated the formalism of vanishing cycles. Let us discuss a very special case when the base space is a punctured disk in order to understand this theory.

1.1. The definition (without monodromy). Let Δ be a small disk in $\mathbb{C}$ centered at 0 with coordinate t, $\Delta^* = \Delta \setminus \{0\}$ the punctured disk. Let $\widetilde{\Delta}^*$ be the universal cover of Δ^*. We choose a coordinate z of $\widetilde{\Delta}^*$, so that the covering map is given by $t = \exp(2\pi i z)$. Let $p\colon \widetilde{\Delta}^* \to \Delta$ be the covering map composed with the inclusion $\Delta^* \to \Delta$, and $i\colon \{0\} \to \Delta$ the inclusion.

For a complex K of sheaves of $\mathbb{C}$-vector spaces on Δ, the *nearby cycle* of K, notation $\mathbb{R}\Psi_t(\mathrm{K})$, is defined as a *pair*

$$\boxed{(i^{-1}\mathbb{R}p_*p^{-1}\mathrm{K}, \mathrm{T})},$$

where the first item is understood as an object of the *derived category* of complexes of sheaves of $\mathbb{C}$-vector spaces on the singleton $\{0\}$ (thus it is just $\mathrm{D}^b(\mathbb{C})$); and T is an endomorphism of the complex $i^{-1}\mathbb{R}p_*p^{-1}\mathrm{K}$ called the *monodromy operator*

In short, the nearby cycle is obtained by the following means:

- restrict to the open part and kill the monodromy,
- push back to the total space and restrict to the special fiber.

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In the present definition, the first point is furnished by taking the complex to the space $\widetilde{\Delta}^*$. Later we will see more "monodromy killers" for regular connections. These different killers will give different presentations of the nearby cycle functor.

Let us postpone the definition of the monodromy operator (which is formal), and try first to understand what does the complex $i^{-1}\mathbb{R}p_*p^{-1}\mathrm{K}$ mean in the case that concerns us, namely, when K is the de Rham complex $DR(\mathrm{M})$ of an integrable connection on Δ^* that is meromorphic at 0. The first functor p^{-1} is easy, as it simply takes the de Rham complex to (recall that $p^{-1} = p^*$ as it is a local isomorphism) the de Rham complex of the inverse image:

$$DR(p^*\mathrm{M}): \quad p^*\mathrm{M} \xrightarrow{\partial_z} p^*\mathrm{M}$$

The second step is the push forward operation. By complex analysis, we know $\mathbb{R}p_*p^*\mathrm{M} = p_*p^*\mathrm{M}$. Thus this second step produces a complex of sheaves whose section over a small disk Δ_ϵ is given by

$$(\mathbb{R}p_*p^{-1}DR(\mathrm{M}))(\Delta_\epsilon): \quad \Gamma(DR(p^*\mathrm{M})|_{\mathrm{Im}\, z>\mathrm{R}}).$$

Finally, taking the inverse image by i is the same as taking the colimit with respect to $\epsilon \to 0$, thus

$$\mathbb{R}\Psi_t(DR(\mathrm{M})) = \operatorname*{colim}_{\mathrm{R}\to+\infty} \Gamma\left(\{z \in \mathbb{C} : \mathrm{Im}\, z > \mathrm{R}\}, DR(p^*\mathrm{M})\right).$$

Observe that this colimit is taking over a *constant* system. Indeed, since $\widetilde{\Delta}^*$ is contractible, $p^*\mathrm{M}$ is a trivial connection, i.e., isomorphic to $(\mathcal{O}^r, d)$. Thus its de Rham complex on $\{z : \mathrm{Im}\, z > \mathrm{R}\}$ reduces to constant sheaf of horizontal sections of $p^*\mathrm{M}$. In particular,

$$(*) \qquad \mathbb{R}\Psi_t(DR(\mathrm{M})) = \mathbb{C}^r_{\bullet}.$$

This agrees with the provisional definition we made before.

Hence, the complex that underlies $\mathbb{R}\Psi_t(DR(\mathrm{M}))$ is just a fancy way to express the space of horizontal sections near a point away from origin.

1.2. Monodromy. Now we turn to monodromy. Let $u\colon z \mapsto z+1$ be the deck transformation of $\widetilde{\Delta}^*$. Then there is a natural isomorphism $u^*p^*\mathrm{K} \to p^*\mathrm{K}$, which by adjunction gives $p^{-1}\mathrm{K} \to u_*p^{-1}\mathrm{K}$. Applying $\mathbb{R}p_*$, using the relation $p \circ u = p$, we get an endomorphism $\mathbb{R}p_*p^{-1}\mathrm{K} \to \mathbb{R}p_*p^{-1}\mathrm{K}$, hence an endomorphism $i^{-1}\mathbb{R}p_*p^{-1}\mathrm{K}$.

Now assume $\mathrm{K} = DR(\mathrm{M})$ as above. How to compute the monodromy matrix in the present situation? The deck transformation acts on the de Rham complex of the inverse image of M. Let $\mathbf{e} = (e_1, \ldots, e_r)$ be a frame of M over Δ^* and $\mathbf{v} = (v_1, \ldots, v_r)$ be the frame of horizontal sections of $p^*\mathrm{M}$. They are related by

$$\boldsymbol{v} = \boldsymbol{e}\mathrm{X}(z).$$

Applying the monodromy operator gives

$$\mathrm{T}(\boldsymbol{v}) = \boldsymbol{e}\mathrm{X}(z+1)$$

This is precisely the definition of monodromy of horizontal sections we discussed before.

Thus we have shown that under the isomorphism $(*)$, the monodromy action on $\mathbb{R}\Psi_t(DR(\mathrm{M}))$ is indeed the monodromy matrix of the space of horizontal sections, which agrees with the provisional definition. Also clear is that the monodromy matrix under $\mathbf{v}$ is $\mathrm{X}(z)^{-1}\mathrm{X}(z+1)$ where X is the coordinate matrix of $\mathbf{v}$ in terms of the single-valued frame $\mathbf{e}$. In particular, $\mathrm{X}(z)^{-1}\mathrm{X}(z+1)$ is an $(r \times r)$-matrix with *constant entries*.

1.3. Moderate nearby cycle. Let M be an integrable connection on Δ^*, meromorphic at 0. Then as we have seen, $\mathbb{R}\Psi_t(DR(\mathrm{M}))$ only depends on the connection $j^{-1}\mathrm{M}$, where $j\colon \Delta^* \to \Delta$ is the open immersion. In the sequel we define a variant of the nearby cycle functor, called the *moderate nearby cycle*, for meromorphic connections. It will turn out that this agrees with the nearby cycle only when the meromorphic connection is regular.

Instead of using all holomorphic functions on the universal covering to calculate the de Rham cohomology, one can instead use a much smaller set of functions, the functions of Nilsson class. Define

$$\mathrm{K}^{\mathrm{mod}} = \left\{ \sum_{\substack{\alpha\in\mathbb{C}\\ p\in\mathbb{N}}} f_{\alpha,p}(z) t^{\alpha} (\log t)^p \,:\, f_{\alpha,p} \in \mathbb{C}\{t\}, \text{finite sum} \right\}.$$

The vector space $\mathrm{K}^{\mathrm{mod}}$ has a natural monodromy action, given by $\log t \mapsto \log t + 2\pi i$.

Then the *moderate nearby cycle* of M is

$$\boxed{\psi_t^{\mathrm{mod}}(\mathrm{M}) = \left[\underset{\bullet}{\mathrm{M}\otimes_{\mathbb{C}\{t\}}\mathrm{K}^{\mathrm{mod}}} \xrightarrow{t\partial_t} \mathrm{M}\otimes_{\mathbb{C}\{t\}}\mathrm{K}^{\mathrm{mod}}\right]}.$$

Here we are using lower case psi to distinguish the present definition, which applies to meromorphic connections, from the previous one, which applies to any complexes of sheaves. The upshot here is that for regular modules, using only moderate functions only can already kill the monodromy.

Theorem. *If* M *is regular at* 0*, then* $\psi_t^{\mathrm{mod}}(\mathrm{M})$ *and* $\mathbb{R}\Psi_t(DR(\mathrm{M}))$ *are quasi-isomorphic.*

Proof. To begin with, there is a natural chain map $\psi_t^{\mathrm{mod}}(\mathrm{M}) \to \mathbb{R}\Psi_t(DR(\mathrm{M}))$, as t^{α} and $(\log t)^p$ are naturally functions on the universal coverings.

Next, we recall the characterization that horizontal sections of M are moderate on any sector. This means that with respect to any frame of M, a basis of multivalued horizontal sections can be found in $\mathrm{M}\otimes_{\mathbb{C}\{t\}}\mathrm{K}^{\mathrm{mod}}$. This shows that the dimension of $\mathrm{H}^0\psi_t^{\mathrm{mod}}(\mathrm{M})$ is the correct one. It remains to show $\mathrm{H}^1\psi_t^{\mathrm{mod}}(\mathrm{M}) = 0$. This is indeed the case: using a horizontal frames, the problem becomes to proving $t\partial_t\colon \mathrm{K}^{\mathrm{mod}} \to \mathrm{K}^{\mathrm{mod}}$ is surjective. But this is very elementary. □

On the other hand, if M is irregular, then $\psi_t^{\mathrm{mod}}(\mathrm{M})$ is not isomorphic to the nearby cycle of $DR(\mathrm{M})$.

Example. The simplest irregular connection already demonstrates this: if M is rank 1 irregular connection $e \mapsto -t^{-2}e$, then the moderate nearby cycle reads

$$\mathrm{K}^{\mathrm{mod}} \xrightarrow{t^2\partial_t - 1} \mathrm{K}^{\mathrm{mod}}.$$

Note that the kernel is zero, as $\exp(-t^{-1})$ is not moderate; the cokernel is nonzero, for

$$t^2 f'(t) - f(t) = -1 \Rightarrow f(t) = \sum_{n=1}^{\infty} (n-1)!t^n,$$

which is not a convergent series.

2. Prepared saturated lattices

The previous section is about "abstract formulas". In practice, what concerns us is the following. Let M be a connection on Δ^*, regular and meromorphic at 0, how do we compute the monodromy matrix of M (i.e., the nearby cycle) *without solving the equation*.

In many circumstances, we know for some reason that a given M is *regular* (this happens for example when M is the restriction of a Gauss–Manin system for a family of algebraic varieties), but the lattice Λ which presents the connection is *not* saturated.

2.1. Saturation. The first step is to replace Λ by its saturation

$$\Lambda_1 = \Lambda_{\mathrm{sat}} = \sum_{i=0}^{\infty} (t\partial_t)^i \Lambda.$$

Then Λ_1 is a saturated lattice (and the sum is a finite one). We have explained this in a previous post.

In particular,

$$\exp(-2\pi i t\partial_t) \colon \Lambda_1/t\Lambda_1 \to \Lambda_1/t\Lambda_1$$

and the monodromy matrix have the same semi-simplification. However, generally the two are not conjugate. We have seen some examples before, here is a more "geometric" example.

Example. Let M be a meromorphic connection of rank 2 defined by a basis $\boldsymbol{e} = (e_1, e_2)$, and

$$t\partial_t \boldsymbol{e} = \boldsymbol{e} \begin{bmatrix} 0 & 2t \\ 2t & -1 \end{bmatrix}.$$

Then $\exp(-2\pi i t\partial_t)$ is the identity operator; but we will see later that the monodromy matrix is not semisimple (the reader can alternatively see this by working out a basis of horizontal sections).

(This example is related to the Bessel function. Geometrically, it is the Fourier transform of the Gauss–Manin system of the algebraic map $x \mapsto x + x^{-1} \colon \mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1$. M is regular at $t = 0$, but is irregular at $t = \infty$.)

It turns out the problem is that the eigenvalues of the residue differ a nonzero integer.

2.2. Prepared lattice. We say a linear operator is *prepared*, if its eigenvalues satisfy $\lambda - \mu \in \mathbb{Z}$ if and only if $\lambda = \mu$. We say a saturated lattice of M is *prepared* if the residue of ∇ with respect to the lattice is prepared.

Some people use the terminology "weakly prepared" instead of "prepared". But "having prepared eigenvalue" is already an awkward sentence; adding "weakly" to it only makes it more creepy. So I decide to ignore this name entirely.

For instance, if the real part of the eigenvalues are in the interval $[0, 1[$, then the operator is prepared.

An alternative way of saying a linear operator is prepared is that there is a section $\tau \colon \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$, such that $\operatorname{Im} \tau$ contains all the eigenvalues.

The following theorem is known since antiquity.

Theorem (Maybe: Fuchs, Frobenius, . . .). *If* Λ *is a prepared saturated lattice of* M*, then the monodromy matrix is conjugate to* $\exp(-2\pi i t\partial_t|_{\Lambda/t\Lambda})$.

Proof. For simplicity I shall prove the theorem assuming the real part of the eigenvalues fall in $[0,1[$, mainly to avoid the awkward section "τ".

2.2.1. Recap: the canonical lattice. To warm up, recall the following construction, which was used in the proof of the "moderate ⇔ regular" theorem. We begin by fixing a $\mathcal{O}[*0]$-basis $\boldsymbol{e}$ of M, then let $\boldsymbol{v} = (v_1, \dots, v_r)$ be a basis of $\mathrm{M}_{\widetilde{\Delta}^*}$ consisting of horizontal sections. Let T be the monodromy matrix with respect to the basis $\boldsymbol{v}$. Then T has a Jordan decomposition $\mathrm{T} = \mathrm{T}_s \cdot \mathrm{T}_u$, with T_s semisimple, T_u unipotent. This allows us to write

$$\mathrm{T}_s = \exp(-2\pi i \mathrm{R}), \quad \mathrm{T}_u = \exp(-2\pi i \mathrm{N})$$

with R semisimple, N nilpotent. Assume that R is chosen so that the real part of its eigenvalues are in $[0,1[$ (in general, choose R so its eigenvalues lie in the image of τ).

Now we define

$$\boldsymbol{e}^{\mathrm{D}} = \boldsymbol{v} \cdot \mathrm{Y}(z), \quad \mathrm{Y}(z) = \exp(2\pi i z\mathrm{R}) \exp(2\pi i z\mathrm{N}).$$

Then

$$\boldsymbol{e}^{\mathrm{D}}(z+1) = \boldsymbol{v}(z+1)\mathrm{Y}(z+1) = \boldsymbol{v}(z)\mathrm{T} \cdot \mathrm{T}^{-1}\mathrm{Y}(z) = \boldsymbol{e}^{\mathrm{D}}(z).$$

Hence, $\boldsymbol{e}^{\mathrm{D}}$ is single valued. Moreover, it is an $\mathcal{O}[*0]$-basis of M, as the transition matrix between $\boldsymbol{e}^{\mathrm{D}}$ and $\boldsymbol{e}$ is moderate, thanks to the regularity of M.

We define Λ^{D} to be the lattice of M generated by $\boldsymbol{e}^{\mathrm{D}}$. It is called the *canonical lattice*.

It is straightforward to calculate the connection matrix under $\boldsymbol{e}^{\mathrm{D}}$:

$$t\partial_t \boldsymbol{e} = \boldsymbol{e}(\mathrm{R} + \mathrm{N}),$$

Thus, the canonical lattice Λ^{D} is prepared, and we can use the residue of the connection with respect to Λ^{D} to calculate the monodromy.

2.2.2. Proof of $\Lambda = \Lambda^{\mathrm{D}}$. We want to show that if Λ is a saturated lattice such that the real part of the eigenvalues of the residue are in $[0,1[$, then $\Lambda = \Lambda^{\mathrm{D}}$. Since we have shown that

$$\exp(-2\pi i t\partial_t|_{\Lambda^{\mathrm{D}}/t\Lambda^{\mathrm{D}}}) = \mathrm{T},$$

proving $\Lambda = \Lambda^{\mathrm{D}}$ will finish the proof of the theorem.

Since Λ and Λ^{D} are two lattices of M, there exists N such that

$$t^{\mathrm{N}}\Lambda^{\mathrm{D}} \subset \Lambda, \quad t^{\mathrm{N}}\Lambda^{\mathrm{D}} \not\subset t\Lambda.$$

Note that $t^m\Lambda^{\mathrm{D}}$ are all saturated lattices of M, and the eigenvalues of the residue with respect to $t^m\Lambda^{\mathrm{D}}$ are in the range $]m-1, m]$.

We have a commutative ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t^{\mathrm{N}+1}\Lambda^{\mathrm{D}} & \longrightarrow & t^{\mathrm{N}}\Lambda^{\mathrm{D}} & \longrightarrow & t^{\mathrm{N}}\Lambda^{\mathrm{D}}/t^{\mathrm{N}+1}\Lambda^{\mathrm{D}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & t\Lambda & \longrightarrow & \Lambda & \longrightarrow & \Lambda/t\Lambda & \longrightarrow & 0 \end{array}$$

which is equivariant with respect to the action of $t\partial_t$. Since $t^{\mathrm{N}}\Lambda^{\mathrm{D}}$ is not contained in $t\Lambda$, the right vertical arrow is nonzero. The image of this map is necessarily stable under $t\partial_t$, hence admits an eigenvector. The real part of the corresponding eigenvalue has to be in the range $[0,1[$, as the image is a subspace of $\Lambda/t\Lambda$. On the other hand, viewing this image as a quotient of $t^{\mathrm{N}}\Lambda^{\mathrm{D}}/t^{\mathrm{N}+1}\Lambda^{\mathrm{D}}$, we infer the real part of the eigenvalue are in $[m, m+1[$. This shows that $m = 0$, i.e., $\Lambda^{\mathrm{D}} \subset \Lambda$. Reversing the role of Λ and Λ^{D} proves $\Lambda = \Lambda^{\mathrm{D}}$, as claimed.

This completes the proof of Theorem 2.2. □

2.3. Shifting a saturated lattice to a prepared one. Given Theorem 2.2, the new problem is how to find a prepared lattice of the regular connection M. Suppose now Λ is a lattice which is saturated by not prepared. Let us explain to how perform a certain construction, classically known as "shearing transformations", to obtain the canonical lattice Λ^{D}. We shall always keep using the example

$$t\partial_t \boldsymbol{e} = \boldsymbol{e}\begin{bmatrix} 0 & 2t \\ 2t & -1 \end{bmatrix}$$

as a guide.

1) Find an integer m such that the residue of the lattice $t^{-m}\Lambda$ has eigenvalues whose real part are all < 1; find a smallest integer ℓ such that the real parts of the eigenvalues of the residue of the lattice $t^{\ell}\Lambda$ are ≥ 1.

In the example, we could take $m = 0$ and $\ell = 2$.

2) Consider the action of $t\partial_t$ on $t^{-m}\Lambda/t^{\ell}\Lambda$. This is an operator acting on a finite dimensional vector space, so we could decompose the $t^{-m}\Lambda/t^{\ell}\Lambda$ into a direct sum of root spaces of $t\partial_t$

$$t^{-m}\Lambda/t^{\ell}\Lambda = \bigoplus_{\alpha} \mathrm{R}_\alpha, \quad \mathrm{R}_\alpha = \{\xi \in t^{-m}\Lambda/t^{\ell}\Lambda : \exists \mathrm{N} (t\partial_t - \alpha)^{\mathrm{N}}\xi = 0\}.$$

In the example, $\Lambda/t^2\Lambda$ is a 4-dimensional vector space, which has a basis $e_1, e_2, e_3 = te_1, e_4 = te_2$, and the matrix of $t\partial_t$ acting on this basis is

$$\begin{bmatrix} 0 & & & \\ & -1 & & \\ & 2 & 1 & \\ 2 & & & 0 \end{bmatrix}.$$

The root space R_0 consists of e_1 and e_4, and the action of $t\partial_t$ on it is not semisimple; the other two root spaces are eigenspaces of -1 and 1 respectively.

3) In the root space decomposition, let $\mathrm{E} = \bigoplus_{\mathrm{Re}\,\alpha \in [0,1[} \mathrm{R}_\alpha$. Let Λ' be the sublattice of $t^{-m}\Lambda$ generated by E.

In the example, Λ' is the one generated by e_1 and te_2. The connection matrix with respect to this new frame is then

$$t\partial_t(e_1, te_2) = (e_1, te_2)\begin{bmatrix} 0 & 2t \\ 2 & 0 \end{bmatrix}.$$

We can then use this residue to get the correct monodromy matrix.

2.4. Nearby cycle and specialization. Let M be an integrable connection on Δ^*, meromorphic and regular at 0. Let Λ^{D} be the canonical lattice of M, such that the eigenvalues of the residue operator have real parts in $[0, 1[$. Then the above discussion implies that the nearby cycle of M can also be defined as

$$\boxed{\psi_t(\mathrm{M}) = \big(\Lambda^{\mathrm{D}}/t\Lambda^{\mathrm{D}}, \exp(-2\pi \mathrm{i}\,\mathrm{Res}_0(\nabla))\big)}.$$

This furnishes our fourth definition of the nearby cycle (for a regular meromorphic connection). In effect, this "calculates" the nearby cycle without solving the equation.

To make the matter fancier, let us consider the t-adic filtration $\{t^m\Lambda^{\mathrm{D}}\}_{m\in\mathbb{Z}}$ of M defined by Λ^{D}. This *is* the toy case of a filtration used by Kashiwara in 1983 to define the nearby cycle functor for $\mathcal{D}$-modules. Malgrange also devised this filtration in 1981 for Gauss–Manin systems. But we would rather reserve the name "Kashiwara–Malgrange filtration" for something more refined.

The associated graded of the t-adic filtration $\bigoplus_{m\in\mathbb{Z}} t^m\Lambda^{\mathrm{D}}/t^{m+1}\Lambda^{\mathrm{D}}$, denoted by $\nu_0(\mathrm{M})$, is naturally a *graded* $\mathbb{C}[t]$-module, hence could be viewed as a quasi-coherent sheaf on the affine line $\mathbb{A}^1$. In a coordinate free fashion, this affine line is the *normal bundle*

$$\mathrm{T}_{\{0\}}\Delta = \operatorname{Spec}\left\{\bigoplus_{m\in\mathbb{N}} t^m\mathcal{O}_{\Delta,0}/t^{m+1}\mathcal{O}_{\Delta,0}\right\}$$

of $\{0\}\in\Delta$. Since the action of t on $\nu_0(\mathrm{M})$ is invertible, $\nu_0(\mathrm{M})$ could be viewed as a sheaf on the punctured normal bundle $(\mathrm{T}_{\{0\}}\Delta)\setminus\{0\}$, which is manifestly *free*. Its rank is equal to the rank of M. Moreover, as ∂_t takes $t^m\Lambda^{\mathrm{D}}$ to $t^{m-1}\Lambda^{\mathrm{D}}$, $\nu_0(\mathrm{M})$ admits a $\mathbb{C}$-linear map

$$\nabla_{\partial_t}\colon \nu_0(\mathrm{M}) \to \nu_0(\mathrm{M})$$

satisfying the Leibniz rule. This equips M an integrable connection on it.

The integrable connection $(\nu_0(\mathrm{M}),\nabla)$ on $\mathrm{T}_{\{0\}}\Delta$ is called the *specialization* of M at 0. Note that the nearby cycle $\psi_t(\mathrm{M})$ can be recovered from $\nu_0(\mathrm{M})$, as it equals the inverse fiber of $\nu_0(\mathrm{M})$ at 1 (indeed at any nonzero point).

The description of $\nu_0(\mathrm{M})$ is rather easy once we know Λ^{D}. Write

$$t^m\Lambda^{\mathrm{D}}/t^{m+1}\Lambda^{\mathrm{D}} = \bigoplus_{\operatorname{Re}\alpha\in[0,1[} \mathrm{R}_{m+\alpha},$$

where R_β is the root space of $t\partial_t$ with respect to the eigenvalue β. Then $\nu_0(\mathrm{M})$ is an infinite direct sum

$$\nu_0(\mathrm{M}) = \bigoplus_{m\in\mathbb{Z}}\left\{\bigoplus_{\operatorname{Re}\alpha\in[0,1[} \mathrm{R}_{m+\alpha}\right\}.$$

The t-action given by $\mathrm{R}_{m+\alpha} = t^m\mathrm{R}_\alpha$.

2.5. The Kashiwara–Malgrange filtration. Keep the notation of the previous paragraph, but I shall assume in addition that the eigenvalue of the monodromy operator of M are of length 1. In other words, the eigenvalues of $t\partial_t$ with respect to any saturated lattice are *real*. This is purely for convenience, so that I don't have to deal with the lexicographic order on $\mathbb{C}$.

With this hypothesis, we can write

$$\Lambda^{\mathrm{D}}/t\Lambda^{\mathrm{D}} = \bigoplus_{\alpha\in[0,1[} \mathrm{R}_\alpha.$$

Recall the definition of the canonical extension. Using the construction, we define $\mathrm{V}^\beta\mathrm{M}$ to be the lattice of M, such that the eigenvalues of the residue $t\partial_t$ are in the interval $[\beta,\beta+1[$. In other words, if $\beta=\beta_0+\lfloor\beta\rfloor$, then

$$\mathrm{V}^\beta\mathrm{M} = \left\{\bigoplus_{\beta_0\le\alpha<1} \mathrm{R}_\alpha\right\} + t^{\lfloor\beta\rfloor+1}\Lambda^{\mathrm{D}} \subset t^{\lfloor\beta\rfloor}\Lambda.$$

Thus $\mathrm{V}^\bullet\mathrm{M}$ is a filtration of M indexed by real numbers. We shall call this filtration the Kashiwara–Malgrange filtration. The point of using V-filtration instead of the t-adic filtration is that V-filtration separates the different eigenvalues of $t\partial_t$. For instance, the unipotent part of the nearby cycle is $\mathrm{Gr}^0_{\mathrm{V}}(\mathrm{M})$.

Definition 1. For $\alpha \in [0,1[$, define

$$\boxed{\psi_{t,\alpha}(\mathrm{M}) = \mathrm{Gr}^{\alpha}_{\mathrm{V}}(\mathrm{M}), \quad \text{equipped with the operator } \exp(-2\pi i t\partial_t)}.$$

Thus $\psi_t(\mathrm{M}) = \bigoplus_{\alpha\in[0,1[} \psi_{t,\alpha}(\mathrm{M})$.

The vector space $\psi_{t,0}(\mathrm{M})$ is called the *unipotent nearby cycle* of M.

We note that the associated graded $\mathrm{Gr}^{\alpha}_{\mathrm{V}}(\mathrm{M})$ admits canonical lifts in M.

Lemma 2. *Define* $\mathrm{M}^{(\alpha)} = \{v \in \mathrm{M}(\Delta) : \exists \mathrm{N}, (t\partial_t - \alpha)^{\mathrm{N}} v = 0\}$. *Then* $\dim \mathrm{M}^{(\alpha)} < \infty$ *and* $\mathrm{M}(\Delta) \subset \prod_{\alpha\in\mathbb{R}} \mathrm{M}^{(\alpha)}$.

Proof. Let $e_1, \ldots, e_r$ be a basis of Λ^{D} such that the matrix of $t\partial$ with respect to $e_1, \ldots, e_r$ is a constant matrix Γ. It is clear that, for $\alpha \in [0,1[$, the vector space

$$\mathrm{M}^{(\alpha)} = \left\{ \sum_{i=1}^{r} x_i e_i \in \mathrm{M}(\Delta) : \exists \mathrm{N}, (\Gamma - \alpha)^{\mathrm{N}} [x_1, \ldots, x_n]^{\mathsf{T}} = 0 \right\}$$

is a finite dimensional subspace of Λ^{D} which splits the space R_α in the quotient $\Lambda^{\mathrm{D}}/t\Lambda^{\mathrm{D}}$. We will see later that this definition of $\mathrm{M}^{(\alpha)}$ agrees with the one defined in the statement of the lemma.

Thus we can write

$$\Lambda^{\mathrm{D}} = \left(\bigoplus_{\alpha\in[0,1[} \mathrm{M}^{(\alpha)} \right) \oplus t\Lambda^{\mathrm{D}}.$$

The same argument actually shows that, for any integers $a < b$, we have

$$t^a \Lambda^{\mathrm{D}} = \left(\bigoplus_{a\le\alpha<b} \mathrm{M}^{(\alpha)} \right) \oplus t^b \Lambda^{\mathrm{D}}.$$

Now each element $v \in \mathrm{M}(\Delta)$ belongs to $t^a\Lambda^{\mathrm{D}}$ for some $a \in \mathbb{Z}$. We have an injective map: $t^a\Lambda^{\mathrm{D}} \to \Lambda^{[a,\infty[}$, where

$$\Lambda^{[a,\infty[} := \lim_{b\to\infty} t^a\Lambda^{\mathrm{D}}/t^b\Lambda \simeq \prod_{a\le\alpha<\infty} \mathrm{M}^{(\alpha)}.$$

From this we see v can be regarded as an element in $\prod_{\alpha\in\mathbb{R}} \mathrm{M}^{(\alpha)}$.

Finally, using the action of $t\partial_t - \alpha$ on the space $\prod_\alpha \mathrm{M}^{(\alpha)}$ we deduce immediately that the $\mathrm{M}^{(\alpha)}$ we considered is the one defined in the statement of the lemma. □

In fact, as one can see from the proof, $\mathrm{M}(\Delta)$ is contained in the "two-sided limit"

$$\Lambda^{]-\infty,\infty[} = \operatorname*{colim}_{a\to-\infty} \lim_{b\to+\infty} t^a\Lambda^{\mathrm{D}}/t^b\Lambda^{\mathrm{D}}.$$

Thus the Kashiwara–Malgrange filtration can be described using $\Lambda^{]-\infty,\infty[}$ as

$$\mathrm{V}^\alpha\mathrm{M}(\Delta) = \mathrm{M}(\Delta) \cap \prod_{\beta\ge\alpha} \mathrm{M}^\beta \subset \Lambda^{]-\infty,\infty[}.$$

To conclude this section, we shall give a different characterization of the root spaces $\mathrm{M}^{(\alpha)}$. Define $\mathcal{J}_{\alpha,m}$ to be the meromorphic connection on Δ^* of rank m defined by

$$t\partial_t(e_1, \ldots, e_m) = (e_1, \ldots, e_m) \cdot \mathrm{J}_{\alpha,m},$$

where $\mathrm{J}_{\alpha,m}$ is the $(m \times m)$-Jordan block with eigenvalue α. The regular meromorphic connections $\mathcal{J}_{\alpha,m}$, as m varies, form an increasing sequence

$$\mathcal{J}_{\alpha,1} \subset \mathcal{J}_{\alpha,2} \subset \cdots \subset \mathcal{J}_{\alpha,m} \subset \cdots$$

such that the quotients are all isomorphic to $\mathcal{J}_{\alpha,1}$. They also form a projective system

$$\mathcal{J}_{\alpha,1} \twoheadleftarrow \mathcal{J}_{\alpha,2} \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{J}_{\alpha,m} \twoheadleftarrow \cdots,$$

since $\mathcal{J}_{\alpha,m}/\mathcal{J}_{\alpha,1} \simeq \mathcal{J}_{\alpha,m-1}$. These are all standard properties of the Jordan block $\mathrm{J}_{\alpha,m}$. The projective system

$$\mathcal{J}_{\alpha} = \text{“}\lim_{m\to\infty}\text{”}\, \mathcal{J}_{\alpha,m}$$

can be interpreted as a differential system defined by an infinite Jordan block.

Lemma 3. *We have*

$$\mathrm{M}^{(\alpha)} = \operatorname*{colim}_{m\to\infty} \mathrm{H}^0_{\mathrm{DR}}(\Delta, \mathcal{H}om_{\mathcal{O}[*0]}(\mathcal{J}_{\alpha,m}, \mathrm{M})),$$

and

$$\operatorname*{colim}_{m\to\infty} \mathrm{H}^1_{\mathrm{DR}}(\Delta, \mathcal{H}om_{\mathcal{O}[*0]}(\mathcal{J}_{\alpha,m}, \mathrm{M})) = \{0\}.$$

Note that the colimits are taken over two essentially constant systems.

The proof is left as an exercise.

From this lemma we get another characterization of the nearby cycle functor: for $\alpha \in [0, 1[$,

$$\boxed{\psi_{t,\alpha}(\mathrm{M}) = i^{-1} DR(\mathcal{H}om(\mathcal{J}_{\alpha}, \mathrm{M}))}.$$

In view of the philosophy that nearby cycle is all about “killing monodromy”, the “infinite Jordan block” $\mathcal{J}_{\alpha}$ in the present description is the monodromy killer.