REGULAR MEROMORPHIC CONNECTIONS ON A DISK

Throughout, Δ will be a small disk in $\mathbf{C}$ (radius not specified) centered at 0, and $\Delta^* = \Delta - \{0\}$. Fix a coordinate t of Δ. We write $\mathcal{O}$ instead of $\mathcal{O}_\Delta$ when there is no ambiguity. Write

$$\mathcal{O}[*0] = \operatorname{colim}\left\{\mathcal{O} \xrightarrow{t} \mathcal{O} \xrightarrow{t} \cdots\right\} = \mathcal{O}[t^{-1}].$$

Thus for $\mathrm{U} \subset \Delta$, $\mathcal{O}[*0](\mathrm{U})$ is the set of holomorphic functions on $\mathrm{U} - \{0\}$ that are meromorphic at 0.

1. Regularity

1.1. Lattices. Let M be a free $\mathcal{O}[*0]$-module on Δ. A *lattice* of M is a finite $\mathcal{O}_\Delta$-submodule Λ of M (necessarily free over $\mathcal{O}$), such that $\Lambda \otimes_{\mathcal{O}} \mathcal{O}[*0] = \mathrm{M}$.

Each $\mathcal{O}[*0]$-basis $(e_1, \dots, e_r)$ of M gives rise to a lattice $\Lambda = \bigoplus_{i=1}^r \mathcal{O}e_i$ of M. Conversely, any lattice Λ possesses a $\mathcal{O}$-basis (which is necessarily also a $\mathcal{O}[*0]$-basis of M).

Two $\mathcal{O}[*0]$-bases $\mathbf{e}^{(1)} = (e_1^{(1)}, \dots, e_r^{(1)})$ and $\mathbf{e}^{(2)} = (e_1^{(2)}, \dots, e_r^{(2)})$ determine the same lattice if and only if there exists a matrix $g \in \mathrm{GL}_r(\mathcal{O}(\Delta))$ such that $\mathbf{e}^{(1)} \cdot g = \mathbf{e}^{(2)}$.

Suppose Λ_1 and Λ_2 are two lattices of M. Then there exists a positive integer N such that $t^{\mathrm{N}}\Lambda_1 \subset \Lambda_2 \subset t^{-\mathrm{N}}\Lambda_1$. This can be seen by "clearing denominators" of the base-change matrices.

1.2. Definition of regularity. Now suppose (M, ∇) is an integrable connection on Δ^*, meromorphic at 0. We say a lattice Λ of M is *saturated*, if $\nabla_{t\partial_t}\Lambda \subset \Lambda$. If M possesses a saturated lattice, we say (M, ∇) is *regular* (or has *regular singularity*) at 0.

Regular connections are among the simplest meromorphic connections, and some most natural connections appearing in algebraic geometry (e.g., the Gauss–Manin connection) are regular.

It could happen that a regular (M, ∇) is presented by a non-saturated lattice. For instance, consider the meromorphic connection

$$t\partial_t(e_1, e_2) = -(e_1, e_2)\begin{bmatrix} 0 & t^{-1} \\ 0 & 1 \end{bmatrix}. \tag{1.2-1}$$

A priori $t\partial_t|_\Lambda$ has a second order pole at 0. But we could use another lattice

$$\Lambda + t\partial_t\Lambda = \operatorname{Span}(u_1, u_2),$$

where $u_2 = e_2, u_1 = t\partial_t(e_2) - e_2 = -t^{-1}e_1$. A direct computation shows that the new connection matrix is given by

$$t\partial_t(u_1, u_2) = -(u_1, u_2)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

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In general, if we already know, for some reason, that M is regular, then we could turn any lattice Λ into a saturated one via the *saturation*

$$\Lambda_{\mathrm{sat}} = \sum_{i=0}^{\infty} (t\partial_t)^i \Lambda. \tag{1.2-2}$$

Plainly Λ_{sat} is stable by $t\partial_t$. To see this is a lattice, i.e., it is finite over $\mathcal{O}$, we use the regularity of M: let Λ' be a saturated lattice of M, then there exists $m \in \mathbf{Z}$ such that

$$\Lambda \subset t^{-m}\Lambda'.$$

As $t^{-m}\Lambda'$ is finite, and as $\mathbf{C}\{t\}$ is noetherian, Λ_{sat} is also finite. The noetherian property actually implies that the infinite sum (1.2-2) is a finite one,

1.3. A criterion of regularity. We give a criterion of regularity based on functions with moderate growth.

Recall the notion of angular sectors. Let $\mathrm{I} \subsetneq \mathbf{S}^1$ be a connected subset of $\mathbf{S}^1$, ($\mathbf{S}^1$ is viewed as the unit circle in the complex plane). Define

$$\Sigma_{r,\mathrm{I}} = \{z \in \Delta^* : \operatorname{Arg} z \in \mathrm{I}, |z| < r\}.$$

Sets of the form $\Sigma_{r,\mathrm{I}}$ are referred to as *angular sectors* of Δ^*. If I is a closed (resp. open, resp. half open) interval of $\mathbf{S}^1$, we say the angular sector $\Sigma_{r,\mathrm{I}}$ is closed (resp. open, resp. half open).

Now we can define the notion of functions with moderate growth. A function $g(t)$ defined on an open subset U of Δ^* is said to have *moderate growth at* 0, if there exists N such that

$$|t|^{\mathrm{N}} \cdot |g(t)|$$

is uniformly bounded on any closed angular sector contained in U.

For example, if U is simply connected, then for any $p \in \mathbf{N}$, any branch of $(\log t)^p$ on U is moderate, so are the branches of t^α on U for any $\alpha \in \mathbf{C}$. On the other hand, $\exp(t^{-1})$ is not moderate on any open subset which intersects any the sectors $\Sigma_{r,[-\frac{\pi}{2},\frac{\pi}{2}]}$; but it is moderate for open subsets contained in $\Sigma_{r,]\frac{\pi}{2},\frac{3\pi}{2}[}$. A holomorphic function on Δ^* is moderate in any angular sector if and only if it is a function meromorphic at 0.

Let us now turn to the criterion of regularity. Let (M, ∇) be an integrable connection on Δ^* that is meromorphic at 0. Let $(e_1, \dots, e_r)$ be an $\mathcal{O}[*0]$-basis of M on Δ^*. Then the action of ∇ can be presented using a matrix A whose entries are functions that are meromorphic at 0:

$$t\partial_t(e_1, \dots, e_r) = (e_1, \dots, e_r) \cdot \mathrm{A}(t).$$

For an open subset U of Δ^*, an r-tuple of holomorphic functions on

$$\boldsymbol{x}(t) = [x_1(t), \dots, x_r(t)]^{\mathsf{T}} \in \mathcal{O}(\mathrm{U})^{\oplus r}$$

thus represents a section $(e_1, \dots, e_r) \cdot \boldsymbol{x}(t)$ of M. We say $\boldsymbol{x}(t)$ has moderate growth if its components have moderate growth. The r-tuple $\boldsymbol{x}(t)$ corresponds to a *horizontal* section of M if and only if it satisfies the following system of ordinary differential equations

$$t\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \mathrm{A}(t)\boldsymbol{x}(t) = 0. \tag{1.3$*$}$$

Theorem. *In the notation above,* M *is regular if and only if the solutions of* (1.3∗) *on any angular sector all have moderate growth.*

The proof will be given at the end of the notes.

Consider as an example again the meromorphic connection defined by

$$t\partial_t(e_1, e_2) = -(e_1, e_2)\begin{bmatrix}0 & t^{-1}\\ 0 & 1\end{bmatrix}$$

(1.2-1). A fundamental solution matrix of the associated system of ordinary differential equations is

$$(e_1, e_2)\begin{bmatrix}\log t & 1\\ t & 0\end{bmatrix},$$

which is moderate on any sector. This shows again that this connection is regular.

An example of an irregular meromorphic connection is given by $t^2\partial_t e = -e$. A horizontal section of this connection is $\exp(t^{-1})\cdot e$. On the sector $\mathrm{Re}(t) > 0$, the function $\exp(t^{-1})$ has exponential growth (*not* moderate); while on the sector $\mathrm{Re}(t) < 0$, it has exponential decay (moderate). This is the simplest instance of the so-called *Stokes phenomenon*.

1.4. Tate's elliptic curve. Integrable connections with regular singular singularities appear naturally in geometry. We demonstrate this point using the so-called Gauss–Manin connection of the "Tate elliptic curve".

Let Δ^* now be the unit punctured disk with coordinate q. Define $\mathrm{X}(q) = \mathbf{C}^*/q^{\mathbf{Z}}$. This is a compact complex manifold homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$. When q varies, the spaces $\{\mathrm{X}(q)\}_{q\in\Delta^*}$ form a "family of elliptic curves" over Δ^*, i.e., there is a complex manifold X^* and a proper holomorphic map $f\colon \mathrm{X}^* \to \Delta^*$ such that $f^{-1}(q) = \mathrm{X}(q)$. Moreover, this family can be "compactified", in that there is a complex manifold X, containing X^* as an open subspace, a proper holomorphic map $\mathrm{X} \to \Delta$, still denoted by f, such that the following diagram

$$\begin{array}{ccc} \mathrm{X}^* & \hookrightarrow & \mathrm{X} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^* & \hookrightarrow & \Delta \end{array}$$

is cartesian.

For each $q \in \Delta^*$, the homology of $\mathrm{X}(q)$ is free of rank 2, with two generators a and b chosen as follows: a is the class of the unit counter-clockwise circle in $\mathbf{C}^*$, b is the image of the path joining 1 and $q \in \mathbf{C}^*$. The cycle b is the so-called "vanishing homology class" of this degeneration; it shrinks to a point as $q \to 0$.

For each q, the space of holomorphic 1-forms on $\mathrm{X}(q)$ is also 1-dimensional, generated by the descendant $\omega(q)$ of the $q^{\mathbf{Z}}$-invariant 1-form $\mathrm{d}z/z$ of $\mathbf{C}^*$.

Using the two homology cycles a and b and the holomorphic 1-form $\omega(q)$, we can form the period integrals

$$\int_a \omega(q) = 2\pi \mathrm{i}, \quad \int_b \omega(q) = \int_1^q \frac{\mathrm{d}z}{z} = \log q.$$

In other words, if we use A, B to denote the dual basis of a, b in cohomology, then $\omega(q) = 2\pi\mathrm{i}\mathrm{A} + (\log q)\mathrm{B}$. The Gauss–Manin connection operates on the cohomology by killing A and B. Thus $\nabla_{q\partial_q}\omega(q) = \mathrm{B}$. It follows that with respect to the basis $\omega(q)$ and

B, the Gauss–Manin connection is presented as

$$\nabla_{q\partial_q}(\mathrm{B},\omega(q)) = (\mathrm{B},\omega(q))\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

This is a connection with regular singularity at $q = 0$.

2. Monodromy

2.1. The monodromy representation. Let (M,∇) be an integrable connection on Δ^* (without meromorphic structure whatsoever). For convenience let us present this connection under an $\mathcal{O}_{\Delta^*}$-basis of M:

$$\nabla_{t\partial_t}(e_1,\dots,e_r) = (e_1,\dots,e_r)\cdot \mathrm{A}(t).$$

where $\mathrm{A}(t)$ is an $(r\times r)$-matrix with entries in $\mathcal{O}(\Delta^*)$, r is the rank of M.

For open subset U of Δ^*, the space of horizontal sections of M over U is precisely the set of r-tuple of holomorphic functions $(x_1(t),\dots,x_r(t))^{\mathsf{T}}$ on U, satisfying the following linear system of ordinary differential equations

$$\begin{bmatrix}t\dot{x}_1(t)\\ \vdots\\ t\dot{x}_r(t)\end{bmatrix} + \mathrm{A}(t)\begin{bmatrix}x_1(t)\\ \vdots\\ x_r(t)\end{bmatrix} = 0.$$

We use the notation $\mathrm{E}(\mathrm{U})$ to denote this $\mathbf{C}$-vector space. By a theorem of Cauchy, when U is simply connected, $\mathrm{E}(\mathrm{U})$ is an r-dimensional $\mathbf{C}$-vector space.

For each $a\in\Delta^*$, the inductive system

$$\Big(\mathrm{E}(\mathrm{U}) : \mathrm{U}\ni a \text{ open}\Big)$$

is eventually constant. The colimit

$$\mathrm{E}_a = \operatorname*{colim}_{\mathrm{U}\ni a}\mathrm{E}(\mathrm{U})$$

is therefore an r-dimensional $\mathbf{C}$-vector space.

The key feature of the assignment $\mathrm{U}\mapsto\mathrm{E}(\mathrm{U})$ is that it satisfies the so-called analytic continuation property. A fancy version of it says that the espace étalé of the sheaf E is a Hausdorff space. In concrete terms, it means the following. Let a and b be two points of Δ^*, and let $\gamma\colon[0,1]\to\Delta^*$ be a path. Then we can cover $\gamma([0,1])$ be finitely small open disks $\mathrm{D}_1,\mathrm{D}_2,\dots,\mathrm{D}_m$ centered at a_i such that $a = a_1$, $b = a_m$, and $\mathrm{D}_i\cap\mathrm{D}_{i-1}\neq\emptyset$ for $i = 2,3,\dots,m$.

By the uniqueness of solutions of ordinary differential equation, the maps

$$\mathrm{E}_{a_i}\leftarrow\mathrm{E}(\mathrm{D}_i)\to\mathrm{E}(\mathrm{D}_i\cap\mathrm{D}_{i-1})\leftarrow\mathrm{E}(\mathrm{D}_{i-1})\to\mathrm{E}_{a_{i-1}}\quad i = 2,3,\dots,m$$

are all isomorphisms of vector spaces. Composing from left to right gives an isomorphism $\mathbb{M}_\gamma\colon\mathrm{E}_a\to\mathrm{E}_b$. This isomorphism depends only on the homotopy class of γ, and not on the specific representative. The uniqueness of the solutions also implies that $\mathbb{M}_{\gamma_1\cdot\gamma_2} = \mathbb{M}_{\gamma_2}\circ\mathbb{M}_{\gamma_1}$. When $a = b$, the map $\gamma\mapsto\mathbb{M}_\gamma$ then gives rise to a linear representation

$$\rho_a\colon\pi_1(\Delta^*,a)\to\mathrm{GL}(\mathrm{E}_a),\quad \rho_a(\gamma) = \mathbb{M}_\gamma.$$

which is called the *monodromy representation* of M.

Concretely, let $\boldsymbol{v}_a$ be a basis of E_a, and γ the class of the circle going counterclockwise around the origin with radius $|a|$. Then there is a matrix T_a such that $\mathbb{M}_\gamma\boldsymbol{v}_a = \boldsymbol{v}_a\cdot\mathrm{T}_a$.

The matrix T_a is called the *monodromy matrix* at a with respect to the horizontal basis $\boldsymbol{v}_a$.

2.2. Monodromy of a constant connection matrix. Consider the integrable connection of rank r defined by

$$t\partial_t(e_1, \dots, e_r) = (e_1, \dots, e_r) \cdot \Gamma, \quad \text{where } \Gamma \in \mathrm{M}_r(\mathbf{C}).$$

Then one checks easily that the coordinates of the horizontal sections are $\mathbf{C}$-linear combinations of the columns of the matrix $t^{-\Gamma}$.

Write $t = \exp(2\pi \mathrm{i} z)$. Then going around a point a counter-clockwise corresponds to the transform $z \mapsto z + 1$. Therefore $\mathtt{M}_\gamma t^\Gamma = t^\Gamma \exp(-2\pi\mathrm{i}\Gamma)$. In other words, its monodromy matrix is $\exp(-2\pi\mathrm{i}\Gamma)$.

The periods of the Tate curve appeared in §1.4 furnishes a geometric example. Using the current notation, a basis of horizontal sections is given by

$$(v_1, v_2) = (e_1, e_2) \begin{bmatrix} 1 & -\log t \\ 0 & 1 \end{bmatrix}.$$

A direct computation gives

$$\mathtt{M}_\gamma(v_1, v_2) = \begin{bmatrix} 1 & -\log t - 2\pi\mathrm{i} \\ 0 & 1 \end{bmatrix} = (v_1, v_2) \begin{bmatrix} 1 & -2\pi\mathrm{i} \\ 0 & 1 \end{bmatrix}.$$

2.3. The nearby cycle functor (provisional version). Let (M, ∇) be an integrable connection on Δ^*, meromorphic at 0. Provisionally, we define the *nearby cycle* of M with respect to the coordinate function t to be the pair

$$\psi_t(\mathrm{M}, \nabla) = (\mathrm{E}_a, \rho_a),$$

where $a \in \Delta^*$ is a point.

Although we have insisted on defining ψ_t for meromorphic connections, it should be clear that $\psi_t(\mathrm{M}, \nabla)$ only depends on the inverse image $\mathrm{M}|_{\Delta^*}$ over Δ^*.

Changing the point a within Δ^* will result an isomorphic vector space, and the monodromy matrix is conjugated. Thus the pair $\psi_t(\mathrm{M}, \nabla)$ is well-defined (independent of a).

In next section, we shall elaborate on the following points: if M is *regular* at 0, then

- the nearby cycle $\psi_t(\mathrm{M}, \nabla)$ could be computed purely algebraically; and
- the isomorphism class of M (as a meromorphic connection) is completely determined by $\psi_t(\mathrm{M}, \nabla)$.

By contrast, $\psi_t(\mathrm{M}, \nabla)$ says nearly nothing when M is not regular. Consider the meromorphic connection defined by $t^2\partial_t e = e$. As we have discussed in §1.3, this connection is irregular, and its horizontal section $\exp(t^{-1})e$ although presents an essential singularity at 0, is single-valued. Thus the monodromy matrix is the identity, and its nearby cycle is isomorphic to $(\mathbf{C}, \mathrm{Id})$, which is also the nearby cycle of the trivial connection. Obviously, M is nontrivial as a meromorphic connection.

2.4. Characteristic polynomials of monodromy. In order to illustrate that regular meromorphic connections are favorable in computing the nearby cycle, let us show how to calculate the *eigenvalues* of the monodromy representation of a regular meromorphic connection *without solving the equation.*

We begin by introducing the notion of the *residue operator* of a saturated lattice.

Let (M, ∇) be a meromorphic connection with regular singularity at 0. Let Λ is a saturated lattice of (M, ∇). The *residue* of ∇ at 0, with respect to Λ, is the linear operator

$$\mathrm{Res}_0(\nabla)\colon \Lambda/t\Lambda \to \Lambda/t\Lambda, \quad [\xi] \mapsto \nabla_{t\partial_t}\xi \mod t\Lambda.$$

If $(e_1, \dots, e_r)$ is an $\mathcal{O}$-basis of Λ, and $\mathrm{A}(t)$ is the matrix of $t\partial_t$, then the matrix of the residue operator with respect to this basis is $\mathrm{A}(0)$. For $a \in \Delta^*$, let $\mathbb{M}$ be the monodromy operator of the counter-clockwise generator of $\pi_1(\Delta^*, a)$.

Theorem. *Let* (M, ∇) *be an integrable connection on* Δ^**, meromorphic and regular at* 0*. Let* Λ *be any saturated lattice of* M*. Then for any* $a \in \Delta^*$*, the characteristic polynomials of* $\mathbb{M}$ *and* $\exp(-2\pi \mathrm{i} \mathrm{Res}_0(\nabla))$ *agree.*

Proof. Let $e_1, \dots, e_r$ be a $\mathcal{O}$-basis of the saturated lattice Λ. Write

$$\nabla_{t\partial_t}(e_1, \dots, e_r) = (e_1, \dots, e_r) \cdot \mathrm{A}(t), \quad \mathrm{A}(t) \in \mathrm{M}_r(\mathbf{C}\{t\}).$$

Introduce the polar coordinate $t = r\exp(\mathrm{i}\theta)$ of Δ^*. Then we have

$$(r\partial_r, \partial_\theta)\begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}^{-1} = (t\partial_t, \bar{t}\partial_{\bar{t}}).$$

The system of ordinary differential equations defining horizontal sections of the connection then translates to

$$\begin{cases} \frac{1}{2}\left(r\partial_r - \mathrm{i}\partial_\theta\right)x + \mathrm{A}(t)x = 0, \\ \frac{1}{2}\left(r\partial_r + \mathrm{i}\partial_\theta\right)x = 0. \end{cases}$$

In particular, when restricting to the circle $r = \rho$ we get an ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\mathrm{i}\mathrm{A}_0 x - \mathrm{i}\rho\mathrm{B}(\theta)x.$$

By the uniqueness of solutions of ordinary differential equations, the monodromy matrix is then the value of fundamental solution matrix of this system of ordinary differential equations at $\theta = 2\pi$. This system depends continuously on the parameter ρ, and when $\rho = 0$ it becomes a system with constant coefficient. A direct integration gives $\exp(-2\pi\mathrm{i}\mathrm{A}_0)$. Since the characteristic polynomial remains constant in this process we see it equals the characteristic polynomial of $\exp(-2\pi\mathrm{i}\mathrm{A}_0)$. □

Consider the system

$$t\partial_t(e_1, e_2) = (e_1, e_2)\begin{bmatrix} 0 & -t \\ 0 & 1 \end{bmatrix}.$$

Then the basis (e_1, e_2) generates a saturated lattice. The matrix $\exp(-2\pi\mathrm{i}\mathrm{Res}_0(\nabla))$ is the identity matrix. Thus, without solving the equation, we know already that the eigenvalues of the monodromy matrix are 1.

However, it should be pointed out that $\exp(-2\pi \mathrm{i} \mathrm{Res}_0(\nabla))$ is in general *not* conjugate to the monodromy matrix. Indeed, in the above example, although $\exp(2\pi \mathrm{i} \mathrm{A}_0)$ is the identity map, $\mathbb{M} \neq \mathrm{Id}$: A direct computation gives a basis of horizontal sections

$$(e_1, e_2)\begin{bmatrix} 1 & \log t \\ 0 & t^{-1} \end{bmatrix}.$$

Therefore the monodromy matrix is given by $\left[\begin{smallmatrix} 1 & 2\pi \mathrm{i} \\ 0 & 1 \end{smallmatrix}\right]$.

3. Characterizing regularity

3.1. An equivalence of category.

Theorem. *The functor* $\mathrm{M} \mapsto j^{-1}\mathrm{M}$ *establishes an equivalence between the category of "connections on* Δ^**, meromorphic and regular at* 0*", and the category of "integrable connections on* Δ^**".*

Proof. That the functor is fully faithful can be translated as follows. Let u be a morphism from $j^{-1}\mathrm{M}$ into $j^{-1}\mathrm{N}$, where M and N are two connections on Δ^*, meromorphic at regular at 0. Then u gives rise to a unique morphism $\mathrm{M} \to \mathrm{N}$

To see this, let A and B be the connection matrix of M and N with respect to their suitable bases respectively, such that A and B have at worst simple poles at 0. Then the morphism u amounts to a matrix satisfying $\mathrm{S}' = \mathrm{SA} - \mathrm{BS}$. By Grönwall's inequality, S has moderate growth at 0, and is therefore meromorphic. This verifies the assertion.

To show that the functor is essentially surjective, Recall that the category of integrable connections on Δ^* is equivalent to the category of pairs (V, T), where V is a finite dimensional complex vector space, and T is a linear operator on V. Fix a matrix Γ such that $\exp(-2\pi\sqrt{-1}\Gamma) = \mathrm{T}$. Then the connection

$$(\dagger) \qquad \nabla \colon \mathrm{V} \otimes_{\mathbf{C}} \mathcal{O}_\Delta[x^{-1}] \to \mathrm{V} \otimes_{\mathbf{C}} \mathcal{O}_\Delta[x^{-1}]\mathrm{d}x, \quad v \otimes 1 \mapsto \Gamma v \otimes \frac{\mathrm{d}x}{x}$$

is regular, and is mapped to (V, T). □

3.2. Proof of Theorem 1.3.

Theorem. *Let* M *be an integrable connection on* Δ^* *meromorphic at* 0*. Then the following are equivalent.*

1) M *is regular at* 0.
2) *There exists an* $\mathcal{O}[*0]$*-basis* $e_1, \dots, e_r$ *of* M *with the following property:*
   - (∗) *For any sector* $\Sigma_{\alpha,\beta}$*, any horizontal section* $s = \sum_{i=1}^r f_i e_i$ *of* M *over* $\Sigma_{\alpha,\beta} - \{0\}$*, the functions* $f_1, \dots, f_r$ *are of moderate growth.*
3) *Any* $\mathcal{O}[*0]$*-basis satisfies the property* (∗).

Proof. (1) ⇒ (2). By Theorem 3.1, if M is regular, there exists an $\mathcal{O}[*0]$-basis under which it has the form (†). With respect to this basis the horizontal sections are linear combinations of the rows of t^Γ, which are of moderate growth. This proves (2).

(2) ⇒ (3) is true because the components of the matrices for changing bases are meromorphic, hence moderate.

(3) ⇒ (1). Let $v_1, \dots, v_r$ be a horizontal basis of $p^{-1}\mathrm{M}$, where $p \colon \widetilde{\Delta}^* \to \Delta^*$ is the universal cover. Let T be the monodromy matrix with respect to the basis $(v_1, \dots, v_r)$.

Define $(e_1, \dots, e_r) = (v_1, \dots, v_r)t^{-\mathrm{T}}$. One checks that $(e_1, \dots, e_r)$ is single-valued, and the matrix of ∇ with respect to $(e_1, \dots, e_r)$ is of the form (†). The only problem is a priori $(e_1, \dots, e_r)$ is only a $\mathcal{O}_{\Delta^*}$-basis of $j^{-1}\mathrm{M}$. To see that it is indeed an $\mathcal{O}[*0]$-basis, one note that the matrix t^{T} is moderate, and therefore for any $\mathcal{O}[*0]$-basis $(e_1', \dots, e_r')$, the changing-basis matrix between $(e_1, \dots, e_r)$ and $(e_1', \dots, e_r')$ is moderate and single-valued. Hence it is an invertible matrix with meromorphic entries. This shows that $(e_1, \dots, e_r)$ is indeed an $\mathcal{O}[*0]$-basis, and it defines a saturated lattice. □