Dwork's analytic continuation theorem revisited

1. In this post, we illustrate how Dwork's theorem, which provides p-adic analytic continuation for some renormalization of Gauß hypergeometric series, follows from some cohomological considerations. The inputs are

   - Berthelot's theorem on overconvergence of Frobenius operations,
   - Dwork's Newton-Hodge decomposition theorem,
   - Katz's correspondence between unit root crystal and p-adic representation of étale fundamental group.

   Dwork proves the existence of analytic continuation by using a complicated congruence relations relations satisfied by the coefficients of the Gauß hypergeometric series. He then uses this analytic continuation to *put Frobenius structure on the Picard-Fuchs equation*. But in this post the order is reversed: we know from Berthelot's theorem that the Frobenius structure exists, and then deduce the existence of the continuation from it. In this cohomological approach, one does not need to know the explicit form of the solution. But this proof should not be thought as a "simple proof" of Dwork's theorem: analytic continuation results of this sort and the existence of Frobenius structure are essentially equivalent, as can be seen from the argument below. So the hidden hard labor is completely due to to Berthelot. (Dwork's theory is reformulated in the crystalline language by Katz.)

   As have said, the existence of these analytic continuations can be explained rather simply: a suitable renormalization of the Gauß hypergeometric series is an incarnation of a Frobenius structure on the Picard-Fuchs module. By Berthelots's theorem, the Frobenius structure is defined globally by analytic functions.

   Although we do use the notion of dagger algebras, we do not really use Monsky-Washnitzer cohomology in any serious fashion. Dagger algebras merely provide a bridge connecting global rings and the Robba ring. This being said, a similar proof also appeared in van der Put's survey on Monsky-Washnitzer cohomology:

   - van der Put, M. (1986). *The cohomology of Monsky and Washnitzer*. Mém. Soc. Math. France (N.S.), (23).

2. **Notations.** Let us consider the Legendre family X of elliptic curves over $\mathbb{Q}_p$. Taking the corresponding rigid analytic space, deleting the tubes around the singular moduli and supersingular moduli, we get an affinoid curve $\mathrm{S} = \mathrm{Sp(A)}$ contained in the closed unit disk. The relative de Rham cohomology and the Gauß-Manin defines a differential A-module $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{X/S})$ with respect to the Fuchsian vector field $\theta = td/dt$. Moreover, the magic of crystalline cohomology equips $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{X/S})$ a Frobenius structure.

   Let $\mathrm{A}^\dagger$ be the "dagger algebra" attached to A consisting of functions that are convergent on both S and some annulus of inner radius $\eta < 1$. Let $\mathscr{R}$ be the Robba ring centered at 0. Let B be the ring of functions that are analytic at 0. So we have a diagram of inclusions

$$\begin{array}{ccc} \mathrm{B} & \longrightarrow & \mathscr{R} \;. \\ \downarrow & & \uparrow \\ \mathrm{A} & \longleftarrow & \mathrm{A}^{\dagger} \end{array}$$

Note that all algebras in the diagram admit Frobenius structure given by $t \mapsto t^p$. It makes sense to talk about (Φ, ∇)-modules on these algebras. The first nontrivial result, due to Berthelot, is the overconvergence of the Frobenius. (The overconvergence of the module and the connection is trivial: they are defined by geometric data which are defined on the whole analytic space $\mathbb{A}^1 \setminus \{0,1\}$.)

Let me now explain why I have to work with the Robba ring. This is because the unique horizontal section of H near 0 can only be defined on the Robba ring, not on B or $\mathrm{A}^{\dagger}$. The point of our discussion is that, nevertheless, some *renormalization* of this horizontal section can be defined on B.

3. **Theorem (Berthelot).** *There exists a* (Φ, ∇)*-module* H *over* $\mathrm{A}^{\dagger}$ *such that* $\mathrm{H} \otimes_{\mathrm{A}^{\dagger}} \mathrm{A} = \mathrm{H}^1_{\mathrm{dR}}(\mathrm{X}/\mathrm{S})$.

   The theorem is announced in the following article as Théorème 5.

   - Berthelot, P. (1983). *Géométrie rigide et cohomologie des variétés algébriques de caractéristique p*. In "Study group on ultrametric analysis, 9th year": 1981/82, No. 3 (Marseille, 1982) (pp. 2–18). Inst. Henri Poincaré, Paris.

   I have not yet found a suitable reference.

4. We shall also use H to denote $\mathrm{H} \otimes_{\mathrm{A}^{\dagger}} \mathscr{R}$. Hopefully this will not cause any confusion. The module H admits a trivialization $\eta, \nabla_{\theta}\eta$, where $\eta = \mathrm{d}x/y$ is the standard 1-form on the projective family $\mathrm{X}/\mathscr{R}$. $\nabla_{\theta}(\eta)$ is the action of Gauß-Manin connection of η.

5. **Descending the unit-root part.** Dwork's contraction mapping argument allows us to extract a (Φ, ∇)-submodule U of $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{X}/\mathrm{S})$, which is isoclinic of slope 0, since we have removed all supersingular disks. Thus by Poincaré duality H has a quotient Q which is isoclinic of slope 1. Moreover, using Cartier operator, one can show that

   - U *is the unique* (Φ,∇)*-submodule of* H *perpendicular to the Hodge bundle* $\mathrm{F}^1\mathrm{H}^1_{\mathrm{dR}}(\mathrm{X}/\mathrm{S})$.

   I now explain how to use Artin-Schreier theory to descend both U and Q to (Φ, ∇)-modules over B.

   Let $f_0 : \mathrm{X}_0 \to \mathbb{A}^1 \setminus \{0,1\}$ be the reduction of the Legendre family. Let E be the restriction of the étale sheaf $\mathrm{R}^1 f_* \mathbb{Z}_p$ to the open subset $\mathrm{S}_0 \subset \mathbb{A}^1 \setminus \{0,1\}$ obtained by removing supersingular points. Then the affinoid space S is the Tubular neighborhood of S_0. Recall that the unit-root part U of the de Rham cohomology $\mathrm{H} \otimes \mathrm{A}$ is equivalent to the lisse étale sheaf E. To prove the extendability, we must prove E is unramified at the point 0. Let us consider the Artin-Schreier sequence

$$0 \to \mathbb{F}_p \to \mathscr{O}_{\mathrm{X}_0} \xrightarrow{\mathrm{F}_{\mathrm{abs}} - \mathrm{Id}} \mathscr{O}_{\mathrm{X}_0} \to 0$$

on X_0. Taking $R^1|_{S_0}$ we get an exact sequence

$$0 \to E \otimes \mathbb{F}_p \to R^1 f_{0*}\mathscr{O}_{X_0} \xrightarrow{F^*_{abs} - Id} R^1 f_{0*}\mathscr{O}_{X_0}$$

The fact that F^*_{abs} on the nodal cubic is nonzero, and the fact that $R^1 f_{0*}\mathscr{O}_{X_0}$ is locally free on $S_0 \cup \{0\}$. This implies that E/pE is unramified at 0. Replacing the Artin-Schreier sequence by its Witt vector version, we conclude that $E/p^n E$, hence E, are all unramified at 0. This shows that U descends to B.

Again, I keep the same notations U and Q for these extended modules. As explained in a letter of mine to A.H., by some simple faithfully flat descent argument, the inclusion $U \to H^1_{dR}(X/S)$ extends to an inclusion of $U \to H$ and remains perpendicular to the Hodge bundle F^1 .

6. Thus, there is an exact sequence

$$0 \to U \to H \to Q \to 0$$

of (Φ, ∇)-modules over $\mathscr{R}$.

However, I shall be working with the dual of H instead of H itself. This is because it is easier to write down the horizontal sections of the dual explicitly.

Let (M, ∇) be the dual (Φ, ∇)-module of H. Let e_1, e_2 be the dual basis with respect to $\eta, \nabla_\theta \eta$. Then with respect to the basis e_1, e_2 the derivation ∇_θ is represented by

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} \theta(v_1) \\ \theta(v_2) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \frac{1}{4(1-t)} & \frac{t}{1-t} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Let $F(t)$ be the Gauß hypergeometric series. Then one can verify easily that

$$s_1 = \begin{bmatrix} F(t) \\ \theta(F(t)) \end{bmatrix}$$

is a *horizontal* section of M over $\mathscr{R}$. Although s_1 does descend to some smaller rings, e.g., it can be defined on the ring $\mathbb{Q}_p[[t]]_0$ of *bounded* analytic functions on the unic disk, it could not be descended to A. This is due to the failure of analytic continuation in p-adic analysis

Passing to the exact sequence $U \to H \to Q$ to the dual, we have an exact sequence

$$0 \to M_{-1} \to M \to M_0 \to 0.$$

Moreover, M inherits a natural Hodge filtration $F^0M \subset F^{-1}M = M$. Since U is perpendicular to F^1H, we conclude that $M_{-1} = Q^\vee$ is perpendicular to F^0M. Since $e_1 = (1,0)^T$ sends η to 1 via the pairing, we conclude that $Q^\vee$ must have a nonwhere vanishing section e of the form $(1, a)^T$, for some $a \in A' \subset \mathscr{R}$, where A' is any localization of A contained in $\mathscr{R}$ on which U is trivialized.

7. **Analytic continuation of** F'/F. Since s_1 is the unique horizontal section of M over $\mathscr{R}$ that descends to $\mathbb{Q}_p[[t]]_0$ (it is even the only formal power series section), the Frobenius action must preserve s_1. Thus $\mathscr{R}s_1$ is a (Φ, ∇)-module of M. Since $F(t)$ is nowhere vanishing, the

subbundle generated by s_1 is perpendicular to F^0M. Since there exists only one (Φ, ∇)-submodule of M perpendicular to F^0M, we infer that we must have $\mathscr{R}s_1 = M_{-1}$.

Thus, we must have $s_1 = \lambda(t)e$. But as the first coordinate of e equals 1, we infer that $\lambda(t) = F(t)$, and $a(t) = \theta(F(t))/F(t)$. This amounts to saying that $\theta(F(t))/F(t)$ admits an analytic continuation to any localization A' of A on which U is trivial. Therefore $\theta(F(t))/F(t)$ is a well-defined analytic function on A. This is the first analytic continuation result, regarding the derivations.

8. **Analytic continuation of** $F(t)/F(t^p)$**.** For anly localization A' of A trivializing U, we write $M_{-1} = A'e$. Since M_{-1} is Frobenius stable, there exists $b(t) \in A'$ such that $\Phi(e) = b(t)e$. Since s_1 is horizontal, $\Phi(s_1)$ is horizontal as well. Thus they differ by a nonzero constant c (Φ being an isomorphism). But then

$$cF(t)e = cs_1 = \Phi(s_1) = \Phi(F(t)e) = F(t^p)b(t)e,$$

we infer that $F(t)/F(t^p) = c^{-1}b(t)$ is analytic in all localization A' of A that trivialize U. In particular, $F(t)/F(t^p)$ must be a global analytic function on $S = Sp(A)$. This is our second analytic continuation result.