1. The *Möbius function* $\mu : \mathbb{Z}_{>0} \to \{0, 1, -1\}$ is defined as follows:

$$\mu(n) = \begin{cases} 1 & n \text{ square free with even prime factors,} \\ -1 & n \text{ square free with odd prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, if n has r distinct prime factors, and the square of each prime factor does not divide n, then $\mu(n) = (-1)^r$; otherwise $\mu(n) = 0$. From the definition it is clear that if m and n are coprime, then $\mu(mn) = \mu(m)\mu(n)$.

An important property of μ is

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases} \tag{1.1}$$

In fact, if $n > 1$ and if $n = p_1^{a_1} \cdots p_\ell^{a_\ell}$ is the prime decomposition of n, then a factor d of n that has potentially nonzero contribution in the displayed sum is of the form

$$p_1^{\epsilon_1} p_2^{\epsilon_2} \cdots p_\ell^{\epsilon_\ell}$$

with ϵ_i equals 0 or 1. Choosing such a d amounts to choosing m primes among the ℓ distinct factors, where there are $\binom{\ell}{m}$ choices, each of which contributes $(-1)^m$ to the sum. In total we have

$$\sum_{d|n} \mu(d) = \sum_{m=0}^{\ell} \binom{\ell}{m} (-1)^m = 0,$$

as desired.

2. Let R be a ring. Let $f \in R[\![T]\!]$ and let $g \in TR[\![T]\!]$. Then it makes sense to talk about the composed power series $f \circ g \in R[\![T]\!]$. Therefore, if R is a ring of characteristic zero. and if we define $\log(1 - T) = -\sum_{m=1}^{\infty} \frac{T^m}{m} \in R[\![T]\!]$, then for each $b \in TR[\![T]\!]$, $\log(1 - b(T))$ is a well-defined formal power series.

3. Let R be a ring. Then it is well-known that $R[\![T]\!]$ is complete with respect to its T-adic topology. Let $f_1, f_2, \dots, f_n, \dots$ be a sequence of power series in $R[\![T]\!]$ such that

$$f_n(T) \in T^n R[\![T]\!]. \tag{3.1}$$

Consider the product

$$F_n(T) = \prod_{i \leq n} (1 + f_i(T)) \in R[\![T]\!].$$

Then we have

$$F_{n+1} \equiv F_n \mod T^n R[\![T]\!]. \tag{3.2}$$

The equation 3.2 means that $\{F_n\}$ is a Cauchy sequence in $R[\![T]\!]$ with respect to the T-adic topology. It follows from the T-adic completeness of $R[\![T]\!]$ that there is a unique $F(T) \in R[\![T]\!]$ such that $F(T) = \lim_n F_n(T)$, and we shall denote this power series $F(T)$ to be the value of the infinite product $\prod_{n=1}^{\infty}(1 + f_n(T))$. Clearly, under the hypothesis (3.1), we have $\prod_{n=1}^{\infty}(1 + f_n(T)) \in 1 + TR[\![T]\!]$.

Example 4. Let R be any ring of characteristic zero. Let us consider the infinite product

$$\prod_{n=1}^{\infty} (1 - T^n)^{-\frac{\mu(n)}{n}} \in 1 + TR[\![T]\!].$$

This infinite product makes sense thanks to the discussion in §3. Taking logarithm, which is eligible by §2, we get an infinite sum

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{T^{mn}}{m} = \sum_{\ell=1}^{\infty} \sum_{d|\ell} \mu(d) \frac{T^\ell}{\ell}$$

Applying (1.1), we conclude that the above sum is simply T. Hence we conclude that

$$\prod_{n=1}^{\infty} (1 - T^n)^{-\frac{\mu(n)}{n}} = \exp(T),$$

where the equality holds on the formal level in $R[\![T]\!]$.

5. Let K be a nonarchimedean field of characteristic 0 whose residue field has characteristic $p > 0$. Let $a \in K$ be an element. The *binomial series* is the formal power series defined by

$$B_a(T) = \sum_{n=0}^{\infty} \binom{a}{n} T^n$$

where

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}.$$

Recall that the radius of convergence ρ of $B_a(T)$ is given by the formula

$$\rho^{-1} = \limsup_{n \to \infty} \left| \binom{a}{n} \right|^{\frac{1}{n}}.$$

If $|a| > 1$, then $|a - i| = |a|$, and we see the upper limit is $|a||p|^{-\frac{1}{p-1}}$, so the radius of convergence of $B_a(T)$ is

$$\rho = |a|^{-1} |p|^{\frac{1}{p-1}} \quad (\text{if } |a| > 1).$$

Now we consider a more specific situation: we assume now $K = \mathbb{Q}_p$. If $a \in \mathbb{Z}$, then the integrality of binomial numbers tells us that $\binom{a}{n} \in \mathbb{Z}_p$. In general by continuity and density of $\mathbb{Z}$ in $\mathbb{Z}_p$, we have the following.

Lemma 6. *If $a \in \mathbb{Z}_p$, then $\binom{a}{n} \in \mathbb{Z}_p$. Therefore $B_a(T) \in \mathbb{Z}_p[\![T]\!]$ if $a \in \mathbb{Z}_p$.* ☐

7. We know that the usual exponential series $\exp(T)$ only converges on the open disk of radius $|p|^{\frac{1}{p-1}}$. In view of the product formula in Example 4 and Lemma 6, it is natural to define a more well-behaved infinite product, the *Artin–Hasse exponential*, to be

$$E_p(T) = \prod_{\substack{n \in \mathbb{Z}_{\geq 1} \\ (n,p)=1}} (1 - T^n)^{-\frac{\mu(n)}{n}}. \tag{7.1}$$

Then using Lemma 6 we know that each factor of the product lies in $\mathbb{Z}_p[\![T]\!]$, hence the infinite product, which makes sens by §3, defines a power series with coefficients in $\mathbb{Z}_p$:

$$E_p(T) \in 1 + T\mathbb{Z}_p[\![T]\!]. \tag{7.2}$$

We would like to relate the Artin–Hasse exponential with the usual exponential. To do this we take the logarithm of $E_p(T)$:

$$\begin{aligned}\log E_p(T) &= \sum_{(n,p)=1} \sum_{m=1}^{\infty} \frac{\mu(n)}{mn} T^{mn} \\ &= \sum_{m=1}^{\infty} \sum_{\substack{d\mid m \\ (d,p)=1}} \frac{\mu(d)}{m} T^{m} \\ &= \sum_{e=1}^{\infty} \frac{T^{p^e}}{p^e}.\end{aligned}$$

The last equality is true, because if $m = p^e m'$, then

$$\sum_{d\mid m,(d,p)=1} \mu(d) = \sum_{d\mid m'} \mu(d) = 0$$

unless $m' = 1$. Therefore we have a formal identity in $\mathbb{Q}_p[\![T]\!]$:

(7.3)

$$E_p(T) = \exp\left(T^p + \frac{T^{p^2}}{p^2} + \cdots + \frac{T^{p^e}}{p^e} + \cdots\right) \in 1 + T\mathbb{Z}_p[\![T]\!].$$