1. Let $(K, |\cdot|)$ be a field with an absolute value. An *extension* of the valued field $(K, |\cdot|)$ is a pair $(K', |\cdot|')$ such that

- K' is a field containing K, and
- $|\cdot|'$ is a nonarchimedean absolute value on K' such that $|a|' = |a|$ for all $a \in K$.

In the sequel, we shall abuse the notations and simply say that K' is a a valued extension of K and we shall keep using $|\cdot|$ to denote the extended absolute value.

2. Radius of convergence. Let K be a valued field, not necessarily complete. Let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ be a formal power series with coefficients in K. The *radius of convergence* ρ of $f(t)$ is then defined to satisfy

$$\rho^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

To justify the name, we must prove the following.

Claim 3. *Let K' be a valued extension of K. Then we have*

(1) *if $\xi \in K'$ and $|\xi| < \rho$, then $\sum a_n \xi^n$ is convergent, and*
(2) *if $\xi \in K'$ and $|\xi| > \rho$, then $\sum a_n \xi^n$ is divergent.*

Proof. Proof of (1). Let $\xi \in K'$ be an element such that $|\xi| < \rho$. Then for n sufficiently large, we have

$$\rho^{-1} \geq |a_n|^{1/n},$$

hence $|a_n|\rho^n \leq 1$. It follows that, for n sufficiently large, we have

$$|a_n \xi^n| = |a_n|\rho^n \left(\frac{|\xi|}{\rho}\right)^n < \left(\frac{|\xi|}{\rho}\right)^n.$$

Since $|\xi|/\rho < 1$, we see $\sum_n a_n \xi^n$ is a convergent sequence. Therefore, $f(t)$ is convergent on the "open" disk $D(0;\rho^-) = \{\xi \in K : |\xi| < \rho\}$.

Proof of (2). If $\xi > \rho$, then we can find a subsequence n_i such that $|a_{n_i}^{1/n_i}|\rho \to 1$, as $i \to \infty$. Thus for i sufficiently large we have $|a_{n_i}|^{1/n_i}\rho \geq \rho/|\xi|$, hence

$$|a_{n_i}\xi^{n_i}| = |a_{n_i}|\rho^{n_i}\left(\frac{\xi}{\rho}\right)^{n_i} \geq 1.$$

as $n_i \to \infty$. This shows that the sum is divergent. □

Aside. The inequality $\rho_1 > \rho_2$ does *not* imply that $D(0;\rho_1^-)$ contains more K-points than $D(0;\rho_2^-)$. For example, if $K = \mathbb{Q}_p$, then $|K| = |p|^{\mathbb{Z}}$. So if we are given an integer $a \in \mathbb{Z}$ and if we choose real numbers ρ_i to satisfy the inequality $|p|^a < \rho_2 < \rho_1 < |p|^{a-1}$, then $D(0;\rho_1^-) = D(0;\rho_2^-)$.

The above assertion works for archimedean as well as nonarchimedean absolute values. The following lemma gives some specific features of the region of convergence in the nonarchimedean case.

Lemma 4. *Let K be a nonarchimedean valued field. Let K' be a valued extension of K. Let $\xi \in K'$. Let $f(t) = \sum_{n=0}^{\infty} a_n t^n \in K[\![t]\!]$ be with radius of convergence ρ. Then the following holds.*

(1) *if $\lim_{n\to\infty} |a_n|\rho^n = 0$, then $f(\xi)$ converges if and only if $|\xi| \leq \rho$,*
(2) *if $|a_n|\rho^n$ does not tend to 0 as n tends to infinity, then $f(\xi)$ converges if and only if $|\xi| < \rho$.*

Proof. Proof of (1). We already know from Claim 3 that $\sum_n a_n \xi^n$ is convergent for all $|\xi| < \rho$. If $|\xi| = \rho$, then condition $\lim_{n\to\infty} |a_n|\rho^n = 0$ is the same as saying $\lim_{n\to\infty} |a_n||\xi|^n = 0$, i.e., $\sum a_n \xi^n$ is convergent.

Proof of (2). If $|a_n|\rho^n$ does not tend to zero, then for all ξ with $|\xi| = \rho$, $\sum a_n \xi^n$ is divergent. So $f(\xi)$ is convergent if and only if $|\xi| < \rho$. □

5. Let K be a nonarchimedean valued field. Let $\rho > 0$ be a real number. Define

$$K\langle T/\rho\rangle = \left\{ \sum_{n=0}^{\infty} a_n T^n \in K[\![T]\!] : |a_n|\rho^n \to 0 \right\}.$$

Then Lemma 4 shows that $K\langle T/\rho\rangle$ agrees with the ring of convergent formal power series on the disk of radius ρ. We shall simply write $K\langle T\rangle$ if $\rho = 1$. The ring $K\langle T\rangle$ is called the ring of restricted power series.

Example 6. For an integer $n \in \mathbb{Z}_{\geq 0}$, write $n = \sum_{j=0}^{\infty} b_j p^j$ with $0 \leq b_j \leq p-1$. Then we have the following formula for the p-adic order of the factorial:

$$\mathrm{ord}_p(n!) = \sum_{j\geq 1} b_j (1 + p \cdots + p^{j-1}).$$

In terms of the p-adic absolute value, we have

$$|n!| = |p|^{\frac{n - \sum_{j=0}^{\infty} b_j}{p-1}}. \tag{6.1}$$

With this estimate at hand we can consider the radius of convergence of the "exponential function" defined by the power series $\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$. We have

$$|n!|^{-1/n} = |p|^{\frac{-1}{p-1}} \cdot |p|^{\frac{1}{n(p-1)}\sum b_j} \leq |p|^{\frac{-1}{p-1}}.$$

On the other hand, taking $n_e = p^e$ we have

$$|n_e!|^{-1/n} = |p|^{\frac{-1}{p-1}} \cdot |p|^{\frac{1}{p^e(p-1)}} \to |p|^{\frac{-1}{p-1}}, \text{ as } e \to \infty.$$

Thus we conclude that $\limsup |n!|^{-1/n} = |p|^{\frac{-1}{p-1}}$. Hence the radius of convergence of $\exp(t)$ is $|p|^{\frac{1}{p-1}}$. Recall that this means that for any valued extension $(K, |\cdot|)$ of $(\mathbb{Q}_p, |\cdot|)$, and any $\xi \in K$

(1) $|\xi| < |p|^{\frac{1}{p-1}}$ implies $\exp(\xi)$ is convergent, and
(2) $|\xi| > |p|^{\frac{1}{p-1}}$ implies $\exp(\xi)$ is divergent.

7. Algebraic theory of substitution. Let R be a ring. Let A be an R-algebra. Then for each $g \in A$ there is a unique homomorphism of R-algebras

$$\mathrm{ev}_g : R[T] \to A,$$

such that $\mathrm{ev}_g(T) = g$. This is learned from any elementary algebra course. In particular, if $A = R[\![T]\!]$, then for each $f \in R[T]$, we denote $\mathrm{ev}_g(f) = f \circ g$, and call this new power series the *composition* of g and f.

However if we replace $R[T]$ by $R[\![T]\!]$, there may or may not exist a well-defined homomorphism ev_g. For instance, if g is the constant power series $g = 1$, and $f(T) = \sum_{n=0}^{\infty} T^n$, then the naive substitution $f(g(T))$ does not make sense.

Note that $R[\![T]\!]$ is the T-adic completion of the polynomial ring $R[T]$. Thus in order to extend map $\mathrm{ev}_g : R[T] \to R[\![T]\!]$ to the power series ring we need to ensure that this it is T-adically continuous. This is the case if and only if $g(T) \in TR[\![T]\!]$.

Thus, as long as the substitute series $g(T)$ is divisible by T, then the power series $g(f(T))$ is well defined.

8. Topological theory of substitution. Now let K be a nonarchimedean field. Given $f(T)$ and $g(T)$ in $K[\![T]\!]$, We would like to ask the convergence of the composition $f(g(T))$. We first need to put some convergence conditions on f and g. Suppose that

- $f \in K\langle T/\rho_1 \rangle$,
- $g \in K\langle T/\rho_2 \rangle$.

We would like to ask when does the formal power series $f(g(T))$ makes sense as a formal power series, adn when is it convergent on the closed disk of radius ρ_2.

Now recall that $K\langle T/\rho \rangle$ is the completion of the polynomial ring $K[T]$ with respect to the ρ-Gauß norm: recall that

$$\left| \sum_{i=0}^{n} a_i T^i \right|_{\rho} = \sup\{|a_i|\rho^i : i = 0, 1, \dots, n\}.$$

Thus, in order to complete the following dashed diagram

(8.1)
$$\begin{array}{ccc} K[T] & & \\ \downarrow & \searrow^{\mathrm{ev}_g} & \\ K\langle T/\rho_1 \rangle & \underset{\varphi}{\dashrightarrow} & K\langle T/\rho_2 \rangle \end{array}$$

we need to ensure that, for each $f(T) = \sum a_i T^i$, we can make the ρ_2-Gauß norm of the the partial sum

$$\left| \sum_{i=m}^{n} a_i g(T)^i \right|$$

arbitrarily small when m is large enough. Note that we have the following simple estimate:

$$\left| \sum_{i=m}^{n} a_i g(T)^i \right| \le \sup \left\{ |a_i|\rho_1^i \cdot \frac{|g(T)|_{\rho_2}^i}{\rho_1^i} \right\}.$$

In the view that $|a_i|\rho_1^i$ is infinitesimal as $i \to \infty$, the above can be made arbitrarily small if $|g|_{\rho_2} \le \rho_1$. Thus we have shown the following.

Lemma 9. *Assume that $g \in K\langle T/\rho_2 \rangle$ is a power series with $|g|_{\rho_2} \le \rho_1$. Then there is a unique dashed arrow φ making the diagram (8.1) commutative. If $g(0) = 0$, then $\varphi(f) = f \circ g$ as power series in $K[\![T]\!]$. Moreover we have, for any $\xi \in K$ with $|\xi| \le \rho_2$, $f(g(\xi)) = \varphi(f)(\xi)$ for all $f \in K\langle T/\rho_1 \rangle$.*

Proof. Besides what have been argued in §8, we still need to equate the value of $f \circ g$ at ξ with the value of f at $g(\xi)$. This amounts to showing the two maps $\mathrm{ev}_\xi \circ \mathrm{ev}_g$ and $\mathrm{ev}_{g(\xi)}$ are the same on $K\langle T/\rho_1 \rangle$. But as they agree trivially on $K[T]$, which is dense in $K\langle T/\rho_1 \rangle$, the assertion follows. □

10. Warning. Without the condition $|g|_{\rho_2} \le \rho_1$, then it could happen that

- $f \circ g$ is convergent at ξ,
- f is convergent at $g(\xi)$, but

- $f(g(\xi)) \neq (f \circ g)(\xi)$.

Below we shall give a famous example.

Example 11. Let $K = \mathbb{Q}_p(\pi)$ where π, the so-called "Dwork pi", is defined to satisfy the equation $\pi^{p-1}+p = 0$. Let $f(T) = \exp(T) = \sum_{n=0}^{\infty} T^n/n!$. Let $g(T) = \pi T - \pi T^p$. Let $E_\pi(T) = (f \circ g)(T)$ be the composition of the two power series. A priori, $E_\pi(T)$ is just a formal power series. But we shall show that actually $E_\pi(T) \in K\langle T\rangle$.

Note that if we were to compute $E_\pi(1)$ we cannot simply plugin $T = 1$ as $|g|_1 = |\pi|$ but $f(T)$ is not in $K\langle T/|\pi|\rangle$. On the other hand, we have $E_\pi(T)^p = (f \circ pg)(T)$. Now $|pg|_1 = |p||\pi|$ and $\exp(T) \in K\langle T/|p||\pi|\rangle$, we can apply Lemma 9 to conclude that

$$E_\pi^p(1) = \exp(p\pi - p\pi) = 1.$$

This means that the power series $E_\pi(T)^p$ is convergent at $T = 1$. This forces the power series $E_\pi(T)$ to be convergent at 1. Therefore we conclude that $E_\pi(T) \in K\langle T\rangle$. However, as $|g|_1$ is bigger than any radius of closed disk on which $f = \exp$ is convergent, we cannot apply Lemma 9 to compute the value of $E_\pi(1)$, as we have said before.

Lemma 12. *Let notations be as in Example 11. We have $E_\pi(1) \neq 1$. Hence $E_\pi(1)$ is a primitive pth root of unity.*

Proof. Note that we have an equality $h(t)\exp(\pi t^p) = \exp(\pi t)$ in the ring of formal power series $K[\![T]\!]$. The coefficients of $\exp(\pi T)$ are $\pi^n/n!$. The absolute value of this is at least π by (6.1). It is equal to $|\pi|$ if and only if $n = p^e$ for $e \geq 1$. Hence $|\exp(\pi T)|_1 = |\pi| = |\exp(\pi T^p)|_1$. This, together with the fact that E_π is convergent at 1, implies that $E_\pi(T) \in \mathcal{O}_K\langle T\rangle$.

The estimate on the coefficients of $\exp(\pi T)$ also gives the following result:

$$\exp(\pi T) \equiv 1 + \sum_{e=1}^{\infty} \pi a_e T^{p^e}$$

with a_e are units in $\mathcal{O}_K$. Write $E_\pi(T) = 1 + \pi h(T)$, we then have

$$\pi h(T) + \sum_{e=1}^{\infty} \pi a_e T^{p^{e+1}} \equiv \sum_{e=1}^{\infty} \pi a_e T^{p^e} \mod \pi^2.$$

Since $h(T)$ is a polynomial modulo π^2, we conclude that the right hand side of

$$\pi g(T) \equiv \pi a_1 T^p + \sum_{e=1}^{\infty} \pi(a_{e+1} - a_e) T^{p^{e+1}} \mod \pi^2.$$

is a polynomial. Thus for N large we have $a_N = a_{N+1} = \cdots = a$. Letting $T = 1$, which is legal, yields that $\pi g(1) = \pi a \neq 0$. Hence we conclude that $E_\pi(1) \equiv 1 + a\pi$ mod π^2. This completes the proof. ∎