

The notations of functors of  $\mathcal{D}$ -modules used in literature are pretty random. Below is a dictionary translating the various notations. The checked ones are the ones I use.

<b>Dictionary</b>			
[HTT08]	[BGK+87]	[BD04]	Bernstein
$\int_f$	$f_+ \checkmark$	$f_+$	$f_*$
$\int_{f!}$	$f! \checkmark$	N/A	$f!$
$Lf^*$	$Lf^\circ$	$f^* \checkmark$	$Lf^\Delta$
$f^\dagger$	$f^! \checkmark$	N/A	$f^!$
$f^*$	$f^+ \checkmark$	N/A	$f^*$

**Warning.** Note that, for a morphism  $f : Y \rightarrow X$ , the functor  $f^*$  defined and used in [BD04, 1.1, first page] equals  $f^![\dim X - \dim Y]$  as defined below.

Let  $f : Y \rightarrow X$  be a morphism of nonsingular algebraic varieties.

- (1) There is the ordinary direct image functor  $f_+$ . For a complex  $M \in D_h^b(\mathcal{D}_Y)$  we have  $f_+M = \mathcal{D}_{X \leftarrow Y} \otimes^L M$ . See [BGK+87, p241].
- (2) There is the ordinary pull-back functor  $f^+$ . This is however defined by  $f^!$  below via duality.
- (3) There is the extraordinary direct image functor  $f_!$ . This is defined by  $f_+$  above via duality.
- (4) There is the extraordinary pull-back functor  $f^!$ . For  $M \in D_h^b(\mathcal{D}_X)$ ,  $f^!M = \mathcal{D}_{Y \rightarrow X} \otimes_{f^{-1}\mathcal{D}_X}^L f^{-1}M[\dim Y - \dim X]$ . See [BGK+87, p234].
- (5)  $f_! \dashv f^!$  and  $f^+ \dashv f_+$ .
- (6) If  $f$  is **proper**, then  $f_! = f_+$ . See [BGK+87, VII.9.12].
- (7) If  $f$  is **smooth**, then  $H^i f^!M = 0$  for all  $i \neq \dim Y - \dim X$ . See [BGK+87, p249, Proposition 4.8]
- (8) If  $f$  is **smooth**, then

$$f^![-2 \dim Y + 2 \dim X] = f^+.$$

See [BGK+87, VII.9.14]

- (9) If  $f$  is a **closed embedding** then  $f_+$  and  $f^!$  define an equivalence between the category of  $\mathcal{D}_Y$ -modules and  $\mathcal{D}_X$ -modules supported on  $Y$ . Moreover  $R\Gamma_{[Y]} = f_+ f^!$ . See [BGK+87, p264, Proposition 7.13].
- (10) **Base change formula.** Given a fiber diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{g} & X \end{array}$$

where all varieties in the diagram are nonsingular. Then for any  $M \in D_h^b(\mathcal{D}_Y)$  we have

$$g^! f_+ M = f'_+ g'^! M,$$

or

$$f'_+ h'^*[\dim X' - \dim X] = g^* f_+[\dim Y' - \dim Y].$$

This is proved in [BGK+87, VI.8.4] assuming that  $g$  is a *locally closed embedding*. A full proof is given in [HTT, Theorem 1.7.3] and [BD04, §A1].

- (11) **Projection formula.** Given  $M \in D_h^b(\mathcal{D}_X)$ , and  $N \in D_h^b(\mathcal{D}_Y)$ , we then have

$$f_+(N \otimes f^!M) = f_+N \otimes M[\dim Y - \dim X]$$

or

$$f_+(N \otimes f^*M) = f_+N \otimes M.$$

See [BGK+87, VII.9.9], [HTT08, Corollary 1.7.5] or [BD04, (1.4)].

- (12) Let  $\mathcal{M}$  and  $\mathcal{N}$  be two complexes of  $\mathcal{D}$ -modules. Define  $\mathcal{M} \otimes^\sim \mathcal{N} = \mathcal{M} \otimes^L \mathcal{N}[-2 \dim X]$ .

- (13) **Fourier transform.** Let  $E$  be a vector bundle on  $X$  with dual bundle  $E^\vee$ . Then for any complex  $\mathcal{M}$  of  $\mathcal{D}_E$ -modules define  $\mathrm{FT}_E(\mathcal{M}) = p_{E^\vee+}(p_E^* \mathcal{M} \otimes \mathrm{can}^* \exp)$ , where
- (a)  $\mathrm{can} : E \times_X E^\vee \rightarrow \mathbb{A}_X^1$  is the natural pairing between  $E$  and its dual,
  - (b)  $\exp$  is the integrable connection on  $\mathbb{A}_X^1$  defined by  $\nabla(e) = e$ ,  $e$  a basis so thus solution of  $\exp$  is  $\exp(-t)$ ,  $t$  being the coordinate on  $\mathbb{A}^1$ , and horizontal sections of  $\exp$  are  $C \exp(t)e$ .
- Fourier transform takes  $\mathcal{D}$ -modules to  $\mathcal{D}$ -modules.

#### REFERENCES

- [BGK+87] — Borel, A., Grivel, P., Kaup, B., Haefliger, A., Malgrange, B., & Ehlers, F. (1987). Algebraic D-modules. Academic Press, Inc., Boston, MA.
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- [HTT08] — Hotta, R., Takeuchi, K., & Tanisaki, T. (2008). D-modules, perverse sheaves, and representation theory. Birkhäuser Boston, Inc., Boston, MA.