On sums of Betti numbers of affine varieties

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Abstract

We show that if V is a subvariety of the affine N-space defined by polynomials of degree at most d, then the sum of its ℓ-adic Betti numbers does not exceed $2(\mathrm{N}+1)^{2\mathrm{N}+1}(d+1)^{\mathrm{N}}$. This answers a question of N. Katz.

1 Introduction

Let V be a finite type, separated scheme over an algebraically closed field k. Katz [Kat01, Theorem 5] proved the existence of a constant $\mathrm{M}(\mathrm{V}/k)$ such that for any prime ℓ invertible in k, the following inequality holds:

$$\mathrm{B}(\mathrm{V})_{(\ell)} := \sum_i \dim \mathrm{H}^i(\mathrm{V}; \mathbf{Q}_\ell) \leqslant \mathrm{M}(\mathrm{V}/k),$$

where $\mathrm{H}^i(\mathrm{V}; \mathbf{Q}_\ell)$ denotes the ℓ-adic cohomology. When k has positive characteristic, it is conjectured that $\mathrm{B}(\mathrm{V})_{(\ell)}$ is independent of ℓ, though it is difficult to prove this in general. Katz's theorem shows that $\mathrm{B}(\mathrm{V})_{(\ell)}$ possesses a uniform upper bound as ℓ varies, which can be interpreted as a weak form of the conjectured independence.

Katz's proof relies on selecting an alteration of V, which leaves the constant $\mathrm{M}(\mathrm{V}/k)$ implicit. He then posed the following question [Kat01, p. 36], paraphrased here:

Question (Katz). *If* $\mathrm{V} \subset \mathbf{A}_k^{\mathrm{N}}$ *is defined by polynomials of degree at most d, can an explicit upper bound for* $\mathrm{M}(\mathrm{V}/k)$ *be provided in terms of d and the number of the defining polynomials?*

If V is *smooth and connected*, Katz [Kat01, Corollary 2] proved that

$$\mathrm{B}(\mathrm{V})_{(\ell)} \leqslant 3 \times 2^r \times (r+1+rd)^{\mathrm{N}},^{1} \tag{K}$$

Thus, the main challenge lies in addressing singular varieties.

When $k = \mathbf{C}$, Milnor [Mil64, Corollary 1] proved

$$\sum_i \dim \mathrm{H}^i(\mathrm{V}^{\mathrm{an}}; \mathbf{Q}) \leqslant d(2d-1)^{2\mathrm{N}-1} \tag{MOT}$$

for the singular cohomology of the complex analytic space V^{an} associated with V. Similar bounds were established by Oleinik [Ole51] and Thom [Tho65]. Since the dimensions of the ℓ-adic cohomology of V and the singular cohomology of V^{an} coincide, this resolves the question for $k = \mathbf{C}$. However, Milnor's proof relies on Morse theory, which does not extend to positive characteristic.

In this brief note, we answer Katz's question with the following theorems:

Theorem 1. *For any algebraically closed field k, any closed subvariety* V *of* $\mathbf{A}_k^{\mathrm{N}}$ *cut out by* $r \geqslant 1$ *polynomials of degree at most d, and any prime ℓ invertible in k, we have*

$$\mathrm{B}(\mathrm{V})_{(\ell)} \leqslant 2r^r(rd+3)^{\mathrm{N}}.$$

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[1] Katz did not focus on obtaining the most optimal bound, making this inequality somewhat crude. His method actually yields a sharper bound: $2^r(rd+r+2)^{\mathrm{N}}$.

If we do not want to specify the number r of defining polynomials of V, we have the following:

Theorem 2. *For any algebraically closed field k, any closed subvariety* V *of* $\mathbf{A}_k^{\mathrm{N}}$ *defined by polynomials of degree at most d, and any prime ℓ invertible in k, we have*

$$\mathrm{B}(\mathrm{V})_{(\ell)} \leqslant 2(\mathrm{N}+1)^{2\mathrm{N}+1}(d+1)^{\mathrm{N}}.$$

In fact, Theorem 2 is a consequence of Theorem 1. We may assume $\mathrm{N} \geqslant 2$ since when $\mathrm{N} = 1$ we trivially have $\mathrm{B}(\mathrm{V})_\ell \leqslant d$. By a classical theorem of Kronecker (cf. [Per42] or [CRW22, §3]), V can always be set-theoretically cut out by at most $\mathrm{N}+1$ nonzero polynomials of degree $\leqslant d$. Hence, by taking $r = \mathrm{N}+1$ in Theorem 1, we get

$$\mathrm{B}(\mathrm{V})_{(\ell)} \leqslant 2 \times [(\mathrm{N}+1)d+3]^{\mathrm{N}} \times (\mathrm{N}+1)^{\mathrm{N}+1} \leqslant 2 \times (\mathrm{N}+1)^{2\mathrm{N}+1} \times (d+1)^{\mathrm{N}}.$$

Remarks. (i) When $k = \mathbf{C}$, Theorem 2 sharpens the classical Milnor–Oleinik–Thom upper bound (MOT) if d is large compared to N^2.

(ii) Consider the number

$$\mathrm{B}(\mathrm{N}, d) \coloneqq \sup \left\{ \mathrm{B}(\mathrm{V})_{(\ell)} : \begin{array}{l} \mathrm{V} = \{f_1 = \cdots = f_r = 0\} \subset \mathbf{A}_k^{\mathrm{N}} \\ \text{for some } r, \text{ with } \deg f_i \leqslant d \end{array} \right\}.$$

If we treat N as fixed and d as a variable, then Theorem 2 implies that $\mathrm{B}(\mathrm{N}, d) \ll_{\mathrm{N}} d^{\mathrm{N}}$. On the other hand, we have $\mathrm{B}(\mathrm{N}, d) \geqslant d^{\mathrm{N}}$. Indeed, if V is a transverse intersection of N sufficiently general degree d polynomials in N variables, then V is a finite set comprised of d^{N} points, and $\mathrm{B}(\mathrm{V})_{(\ell)} = d^{\mathrm{N}}$. Thus $\mathrm{B}(\mathrm{N}, d)$ and d^{N} have the same asymptotic order as $d \to \infty$: $\mathrm{B}(\mathrm{N}, d) \asymp_{\mathrm{N}} d^{\mathrm{N}}$.

(iii) Let R be any commutative ring, and let $\mathcal{V}$ be a finite type separated scheme over R. For any geometric point $x\colon \operatorname{Spec} k \to \operatorname{Spec} \mathrm{R}$, Katz showed that

$$\mathrm{B}(\mathcal{V} \otimes_{\mathrm{R},x} k) \leqslant \mathrm{M}(\mathcal{V} \otimes_{\mathrm{R},x} k/k).$$

However, these upper bounds may depend on the specific geometric point x. In contrast, if

$$\mathcal{V} = \operatorname{Spec} \mathrm{R}[x_1, \dots, x_{\mathrm{N}}]/(f_1, \dots, f_r),$$

with $\deg f_i \leqslant d$, then the bounds established in Theorems 1 and 2 are given by explicit constants that apply uniformly to every geometric point x of Spec R. Thus, these bounds have the advantage of being *uniform in k*.

(iv) Although we have established that d^{N} is the correct asymptotic order of $\mathrm{B}(\mathrm{N}, d)$ as $d \to \infty$, the coefficient $2(\mathrm{N}+1)^{2\mathrm{N}+1}$ of d^{N} in Theorem 2 is far from optimal. There should be considerable room for improvement.

For complete intersections, a slightly sharper bound can be established.

Theorem 3. *Let k be an algebraically closed field. Suppose* $\mathrm{V} \subset \mathbf{A}_k^{\mathrm{N}}$ *is the common zero locus of r polynomials of degree at most d. If* V *has at worst local complete intersection singularities, e.g., if* $\dim \mathrm{V} = \mathrm{N} - r$, *then*

$$\mathrm{B}(\mathrm{V})_{(\ell)} \leqslant 2^r (rd + r + 2)^{\mathrm{N}}.$$

2 The proofs

In the following, we fix an algebraically closed field k and a prime ℓ invertible in k. All schemes are defined over k. We shall write $\mathrm{B}(\mathrm{V})$ instead of $\mathrm{B}(\mathrm{V})_{(\ell)}$.

Suppose $\mathrm{B}(\mathrm{N}, r, d)$ is a positive integer satisfying the following property.

If V *is a closed subscheme of* $\mathbf{A}^{\mathrm{N}}$ *defined by* r *polynomials* $f_1, \ldots, f_r$ *with* $\deg f_i \leqslant d$*, then* $\mathrm{B}(\mathrm{V}) \leqslant \mathrm{B}(\mathrm{N}, r, d)$.

It is not immediately evident that a finite $\mathrm{B}(\mathrm{N}, r, d)$ even exists. This does not follow directly from Deligne's theorem on the constructibility of the direct image of constructible sheaves, since the non-proper direct image does not generally commute with base change. Some additional argument is indeed required, though it is not difficult. We will not dwell on this point, as it will follow from the argument presented below.

Suppose $\mathrm{E}(\mathrm{N}, r, d)$ is a positive integer satisfying the following property:

If V *is a subvariety of* $\mathbf{A}^{\mathrm{N}}$ *defined by nonzero polynomials* $f_1, \ldots, f_r$ *with* $\deg f_i \leqslant d$*, then* $|\chi(\mathrm{V}; \mathbf{Q}_\ell)| \leqslant \mathrm{E}(\mathrm{N}, r, d)$.

Here $\chi(\mathrm{V}; \mathbf{Q}_\ell)$ is the Euler characteristic $\chi(\mathrm{V}; \mathbf{Q}_\ell) = \sum_i (-1)^i \dim \mathrm{H}^i(\mathrm{V}; \mathbf{Q}_\ell)$. By [Lau81], the Euler characteristic equals the compactly supported Euler characteristic: $\chi(\mathrm{V}; \mathbf{Q}_\ell) = \sum_i (-1)^i \dim \mathrm{H}^i_c(\mathrm{V}; \mathbf{Q}_\ell)$. Hence, by [AS88, Theorem 5.27] (and the comment at the beginning of [Kat01, p. 30]) we can take

$$\mathrm{E}(\mathrm{N}, r, d) = 2^r \times (r + 1 + rd)^{\mathrm{N}}. \tag{AS}$$

Upper bound for local complete intersections: Katz's method

Let $f_1, \ldots, f_r \in k[x_1, \ldots, x_{\mathrm{N}}]$ be polynomials of degree $\leqslant d$. Consider the affine scheme

$$\mathrm{V} := \operatorname{Spec} k[x_1, \ldots, x_{\mathrm{N}}]/(f_1, \ldots, f_r).$$

We assume that V is a *set-theoretic local complete intersection*, meaning there is a closed subscheme W of $\mathbf{A}^{\mathrm{N}}$ that is a local complete intersection, with $\mathrm{W}^{\mathrm{red}} = \mathrm{V}^{\mathrm{red}}$. This is the case, for example, if $\dim \mathrm{V} = \mathrm{N} - r$, or if V is smooth. We seek an explicit upper bound for $\mathrm{B}(\mathrm{V})$.

Proposition 1. *In the above situation, we have*

$$\mathrm{B}(\mathrm{V}) \leqslant \mathrm{E}(\mathrm{N}, r, d) + 2 \sum_{i=1}^{\dim \mathrm{V}} \mathrm{E}(\mathrm{N} - i, r, d).$$

In particular, we can take $\mathrm{B}(\mathrm{N}, 1, d)$ *to be* $\mathrm{E}(\mathrm{N}, 1, d) + 2 \sum_{i=1}^{\mathrm{N}} \mathrm{E}(i, 1, d)$.

The proof of Proposition 1 closely follows Katz's proof of inequality (K). Instead of using the standard weak Lefschetz theorem for smooth varieties as Katz did, we apply the following weak Lefschetz theorem for perverse sheaves, due to Deligne.

Theorem (Deligne [Kat93, Corollary A.5])**.** *Let* $\pi \colon \mathrm{X} \to \mathbf{P}^{\mathrm{N}}$ *be a quasi-finite morphism to a projective space. Let* $\mathcal{P}$ *be a perverse sheaf on* X*. Then for a sufficiently general hyperplane* A*, the restriction morphism*

$$\mathrm{H}^i(\mathrm{X}; \mathcal{P}) \to \mathrm{H}^i(\pi^{-1}\mathrm{A}; \mathcal{P}|_{\pi^{-1}\mathrm{A}})$$

is injective if $i = -1$*, and bijective if* $i < -1$.

We now use Deligne's theorem and Katz's original Euler characteristic argument [Kat01, p. 33] to prove Proposition 1.

Proof of Proposition 1. We proceed by induction on the dimension of V. In the base case where $\dim \mathrm{V} = 0$, the result is trivial. Now assume $\dim \mathrm{V} > 0$. Since V is a set-theoretic local complete intersection, the shifted constant sheaf $\mathbf{Q}_{\ell,\mathrm{V}}[\dim \mathrm{V}]$ is a perverse sheaf on V, see e.g., [KW01, Lemma III.6.5]. Now apply Deligne's theorem by

- taking $\mathrm{X} = \mathrm{V}$,

- letting π be the composition of the inclusion map $\mathrm{V} \hookrightarrow \mathbf{A}^{\mathrm{N}}$ and the standard embedding $\mathbf{A}^{\mathrm{N}} \hookrightarrow \mathbf{P}^{\mathrm{N}}$, and
- setting $\mathcal{P} = \mathbf{Q}_{\ell,\mathrm{V}}[\dim \mathrm{V}]$.

Deligne's theorem then implies that for a general affine hyperplane $\mathrm{A} \subset \mathbf{A}^{\mathrm{N}}$, the restriction map

$$\mathrm{H}^i(\mathrm{V}; \mathbf{Q}_\ell) \to \mathrm{H}^i(\mathrm{A} \cap \mathrm{V}; \mathbf{Q}_\ell)$$

is injective when $i = \dim \mathrm{V} - 1$, and bijective if $i < \dim \mathrm{V} - 1$. Ergo,

$$\begin{aligned}
&\dim \mathrm{H}^{\dim \mathrm{V}}(\mathrm{V}; \mathbf{Q}_\ell) \\
&= (-1)^{\dim \mathrm{V}} \chi(\mathrm{V}; \mathbf{Q}_\ell) + \dim \mathrm{H}^{\dim \mathrm{V}-1}(\mathrm{V}; \mathbf{Q}_\ell) - \dim \mathrm{H}^{\dim \mathrm{V}-2}(\mathrm{V}; \mathbf{Q}_\ell) + \cdots \\
&\leqslant (-1)^{\dim \mathrm{V}} \chi(\mathrm{V}; \mathbf{Q}_\ell) + \dim \mathrm{H}^{\dim \mathrm{V}-1}(\mathrm{A} \cap \mathrm{V}; \mathbf{Q}_\ell) - \dim \mathrm{H}^{\dim \mathrm{V}-2}(\mathrm{A} \cap \mathrm{V}; \mathbf{Q}_\ell) + \cdots \\
&= (-1)^{\dim \mathrm{V}} \chi(\mathrm{V}; \mathbf{Q}_\ell) + (-1)^{\dim \mathrm{V}-1} \chi(\mathrm{A} \cap \mathrm{V}; \mathbf{Q}_\ell) \\
&\leqslant \mathrm{E}(\mathrm{N}, r, d) + \mathrm{E}(\mathrm{N}-1, r, d).
\end{aligned} \tag{1}$$

Here we have used the Artin vanishing theorem, which asserts that for a finite type affine scheme V over k, $\mathrm{H}^i(\mathrm{V}; \mathbf{Q}_\ell) = 0$ unless $0 \leqslant i \leqslant \dim \mathrm{V}$.

Note that $\mathrm{A} \cap \mathrm{V}$ is a closed subscheme of the $(\mathrm{N}-1)$-dimensional affine space A, defined by polynomials of degree $\leqslant d$. Since A is generic and V is a set-theoretic local complete intersection, it follows that $\mathrm{A} \cap \mathrm{V}$ also remains a set-theoretic local complete intersection. By inductive hypothesis, we have

$$\mathrm{B}(\mathrm{A} \cap \mathrm{V}) \leqslant \mathrm{E}(\mathrm{N}-1, r, d) + 2 \sum_{i=1}^{\dim \mathrm{V}-1} \mathrm{E}(\mathrm{N}-1-i, r, d). \tag{2}$$

Therefore,

$$\begin{aligned}
\mathrm{B}(\mathrm{V}) &\leqslant \dim \mathrm{H}^{\dim \mathrm{V}}(\mathrm{V}; \mathbf{Q}_\ell) + \mathrm{B}(\mathrm{A} \cap \mathrm{V}) && \text{(Deligne's theorem)} \\
&\leqslant \mathrm{E}(\mathrm{N}, r, d) + \mathrm{E}(\mathrm{N}-1, r, d) \\
&\quad + \mathrm{E}(\mathrm{N}-1, r, d) + 2 \sum_{1 \leq i \leq \dim \mathrm{V}-1} \mathrm{E}(\mathrm{N}-1-i, r, d) && \text{(By (1) and (2))} \\
&= \mathrm{E}(\mathrm{N}, r, d) + 2 \sum_{i=1}^{\dim \mathrm{V}} \mathrm{E}(\mathrm{N}-i, r, d).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3. We take $\mathrm{E}(\mathrm{N}, r, d)$ as in (AS). When $\mathrm{N} = \dim \mathrm{V}$, we have $\mathrm{V} = \mathbf{A}^{\mathrm{N}}$, and the result is trivial. When $\mathrm{N} = 1$, and $\dim \mathrm{V} = 0$, the result is equally trivial since we have $\mathrm{B}(\mathrm{V}) \leq d$. Assume now $\mathrm{N} \geq 2$, and $\dim \mathrm{V} < \mathrm{N}$. Then by Proposition 1, we have

$$\begin{aligned}
\mathrm{B}(\mathrm{V}) &\leqslant 2^r \left[(rd + r + 1)^{\mathrm{N}} + 2 \sum_{i=1}^{\dim \mathrm{V}} (rd + r + 1)^{\mathrm{N}-i} \right] \\
&\leqslant 2^r (rd + r + 2)^{\mathrm{N}}
\end{aligned} \tag{3}$$

thanks to the binomial theorem. This completes the proof. □

Upper bounds in general

Now suppose we have inductively constructed $\mathrm{B}(\mathrm{N}, 1, d)$, $\mathrm{B}(\mathrm{N}, 2, d)$, ..., up to $\mathrm{B}(\mathrm{N}, r-1, d)$, and these numbers are all finite. We proceed to deal with varieties defined by r equations.

Suppose $f_1, \ldots, f_r \in k[x_1, \ldots, x_{\mathrm{N}}]$, $\deg f_i \leqslant d$. Define:

- $F_i \coloneqq \{f_i = 0\}$,
- $W \coloneqq \bigcup F_i$,
- for each $J \subset \{1, \dots, r\}$, $F_J \coloneqq \bigcap_{j \in J} F_j$,
- $V \coloneqq \{f_1 = \cdots = f_r = 0\} = F_{\{1,\dots,r\}}$.

We will bound $B(V)$.

Proposition 2. *In the situation above, we have*

$$B(V) \leqslant B(N, 1, rd) + \sum_{i=1}^{r-1} \binom{r}{i} B(N, i, d).$$

In particular, we can take $B(N, r, d)$ *to be* $B(N, 1, rd) + \sum_{i=1}^{r-1} \binom{r}{i} B(N, i, d)$.

Proof. There is a Mayer–Vietoris spectral sequence for the finite closed covering $\bigcup_{i=1}^{r} F_i$ of W:

$$E_1^{p,q} = \bigoplus_{\mathrm{Card}\, J = p+1} H^q(F_J; \mathbf{Q}_\ell) \Rightarrow H^{p+q}(W; \mathbf{Q}_\ell).$$

We have $E_1^{r-1,q} = H^q(V; \mathbf{Q}_\ell)$. Because $E_\infty^{r-1,q}$ is a subquotient of $H^{q+r-1}(W; \mathbf{Q}_\ell)$, we have

$$\sum_q \dim E_\infty^{r-1,q} \leqslant B(W).$$

For each q and each i, $E_i^{r-1,q}$ appears at the rightmost column of the E_i-page of the spectral sequence, i.e., $E_i^{p,q} = 0$ for $p \geqslant r$. Therefore, we have $E_{i+1}^{r-1,q} = E_i^{r-1,q} / d_i(E_i^{r-1-i,q+i-1})$, where $d_i \colon E_i^{p,q} \to E_i^{p+i,q-i+1}$ is the differential of the spectral sequence. It follows that

$$\begin{aligned} B(W) &= \sum_{p,q} \dim E_\infty^{p,q} \geqslant \sum_q \dim E_\infty^{r-1,q} \\ &= \sum_q \dim E_1^{r-1,q} - \sum_q \sum_{i=1}^{r-1} \dim d_i(E_i^{r-1-i,q+i-1}) \\ &\geqslant B(V) - \sum_{i=1}^{r-1} \sum_q \dim E_1^{r-1-i,q+i-1} \end{aligned} \tag{4}$$

The last inequality holds because for any $i \geqslant 1$, $E_i^{p,q}$ is a subquotient of $E_1^{p,q}$.

For each $p \geqslant 0$, $E_1^{p,q}$ is a direct sum of $H^q(F_J)$ with $\mathrm{Card}\, J = p + 1$. There are $\binom{r}{p+1}$ such summands. Since $F_J \subset \mathbf{A}^N$ is cut out by $p + 1$ polynomials of degree $\leqslant d$, we have

$$\begin{aligned} \sum_q \dim E_1^{p,q} &= \sum_q \bigoplus_{\mathrm{Card}\, J = p+1} H^p(F_J; \mathbf{Q}_\ell) \\ &\leqslant \binom{r}{p+1} B(N, p+1, d). \end{aligned}$$

Hence,

$$\sum_{i=1}^{r-1} \sum_q \dim E_1^{r-1-i,q+i-1} \leqslant \sum_{i=1}^{r-1} \binom{r}{r-i} B(N, r-i, d). \tag{5}$$

Since W is a hypersurface cut out by $f_1 \cdots f_r$, a polynomial of degree $\leqslant rd$, we conclude from Proposition 1 that $B(W) \leqslant B(N, 1, rd)$. Thus Equations (4) and (5) imply that

$$B(V) \leqslant B(N, 1, rd) + \sum_{i=1}^{r-1} \binom{r}{i} B(N, i, d).$$

This completes the proof. □

Proof of Theorem 1. In view of Proposition 1 and Proposition 2, if the numbers $\mathrm{B}(\mathrm{N}, r, d)$ are defined inductively as

- $\mathrm{B}(\mathrm{N}, 1, d) = \mathrm{E}(\mathrm{N}, d) + 2\sum_{i=1}^{\mathrm{N}-1} \mathrm{E}(i, d)$,
- $\mathrm{B}(\mathrm{N}, r, d) = \mathrm{B}(\mathrm{N}, 1, rd) + \sum_{i=1}^{r-1} \binom{r}{i}\mathrm{B}(\mathrm{N}, i, d)$,

where $\mathrm{E}(\mathrm{N}, d)$ is given by (AS), then $\mathrm{B}(\mathrm{V}) \leqslant \mathrm{B}(\mathrm{N}, r, d)$. Let us prove $\mathrm{B}(\mathrm{N}, r, d) \leqslant 2r^r(rd+3)^{\mathrm{N}}$ by induction. For the base case, we invoke Equation (3) with $r = 1$. When $r > 1$, we have

$$\begin{aligned} \mathrm{B}(\mathrm{N}, r, d) &\leqslant 2(rd+3)^{\mathrm{N}} + \sum_{i=1}^{r-1} \binom{r}{i} \times 2i^i(id+3)^{\mathrm{N}} && \text{(inductive hypothesis)} \\ &\leqslant 2(rd+3)^{\mathrm{N}} \left[1 + \sum_{i=1}^{r-1} \binom{r}{i}(r-1)^i\right] && (\text{since } i \leqslant r-1 < r) \\ &\leqslant 2 \times (rd+3)^{\mathrm{N}} \times r^r && \text{(binomial theorem).} \end{aligned}$$

This completes the proof of Theorem 1. □

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