SOME BASIC CONCEPTS

Contents

References.

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- P. Deligne. *Équations différentielles à points singuliers réguliers* (1970). Chapitre I.
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1. Local systems

1.1. The definition of a local system. Let X be a topological space. A sheaf E of complex vector spaces on X is called a *local system* (of complex vector spaces), if X admits an open covering $(\mathrm{U}_i)_{i\in\mathrm{I}}$, such that $\mathrm{E}|_{\mathrm{U}_i}$ is isomorphic to the constant sheaf $\mathbf{C}^n_{\mathrm{U}_i}$ ($n \in \mathbf{N}$). In other words, local system is just another name for a locally constant sheaf valued in some complex vector space. A local system E is said to be *trivial*, if it is a *constant* sheaf of complex vector spaces.

In these notes there are two sources of local systems: those from cohomologies of locally topologically trivial fibrations, and those from linear partial differential equations. The precise content of the first point is: if $f\colon \mathrm{X} \to \mathrm{B}$ is a continuous map which is locally a topologically trivial fibration, then the sheaf $\mathrm{R}^i f_* \mathbf{C}_\mathrm{X}$ associated with the presheaf $\mathrm{U} \mapsto \mathrm{H}^i(f^{-1}(\mathrm{U}), \mathbf{C})$ is a local system on B.

To explain the second point, let X be an open subset of $\mathbf{C}$, let $\mathrm{A}(t)$, be an $(r\times r)$-matrix of holomorphic functions on X. Form the system ordinary differential equations

$$\frac{\partial \boldsymbol{x}}{\partial t} + \mathrm{A}(t)\boldsymbol{x}(t) = 0, \quad \boldsymbol{x}(t) \in \mathscr{O}(\mathrm{X})^{\oplus r}.$$

If $\mathrm{U} \subset \mathrm{X}$ is open, define E(U) be the vector space of holomorphic solutions of this system. Then E(U) is a local system on X.

For if $\Delta \subset \mathrm{U}$ is a sufficiently small disk, a theorem of Cauchy states that the system admits r linearly independent holomorphic solutions over Δ (a fancier version of this is Theorem 2.2). The restriction of these solutions to any smaller open subset of Δ gives a fundamental system of solutions of the system over that smaller open. Whence $\mathrm{U} \mapsto \mathrm{E}(\mathrm{U})$ is locally constant.

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1.2. An example. To be concrete, consider the spaces $\mathrm{X} = \mathbf{C}^2 - \{(x,y) : y^2 = x^3\}$, $\mathrm{B} = \mathbf{C} - \{0\}$, and the map $f(x,y) = y^2 - x^3$. For $t \in \mathbf{C}$, let $\mathrm{X}(t) = \{(x,y) : y^2 = x^3 + t\}$. Thus $f^{-1}(t) = \mathrm{X}(t)$ for $t \neq 0$.

Some observations about the family $\{\mathrm{X}(t)\}_{t \in \mathbf{C}^*}$ are in order.

1) For whatever $t \neq 0$, X(1) is biholomorphic to X(t), an isomorphism is given by

$$\kappa \colon \mathrm{X}(t) \ni (x,y) \mapsto (t^{-1/3}x, t^{-1/2}y) \in \mathrm{X}(1).$$

Note that the definition of κ requires a choice of a third root of t, and a square root of t; thus there are in total 6 such isomorphisms. Indeed, this shows that the family $\{\mathrm{X}(t)\}_{t \in \mathbf{C}^*}$ is *analytically locally trivial*: on any simply connected open subset U of $\mathbf{C}^*$, we can choose uniform branches of the functions $t^{1/3}$ and $t^{1/2}$, and the map

$$\mathrm{X}(1) \times \mathrm{U} \to \mathrm{X} \times_{\mathbf{C}^*} \mathrm{U} = f^{-1}(\mathrm{U}), \quad ((x,y),t) \mapsto (t^{-1/3}x, t^{-1/2}x, t)$$

furnishes a fiber-preserving isomorphism of complex manifolds over U.

This isomorphism can only be made over open subsets on which $t^{1/2}$ and $t^{1/3}$ are uniform; thus it is not unreasonable to speculate that the family $\{\mathrm{X}(t)\}_{t \in \mathbf{C}^*}$ can not be trivialized globally. To see this we need some information about the topology of X(t).

2) The projective completion of X(t) is

$$\overline{\mathrm{X}}(t) = \{[x : y : z] \in \mathbf{P}^{2,\mathrm{an}}_{\mathbf{C}} : x^3 + tz^3 - y^2 z = 0\}$$

which intersects with the line at infinity defined by $z = 0$ at a single point $[0:1:0]$. In particular, X(t) is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$ with a point removed.

3) The fibration $f \colon \mathrm{X} \to \mathrm{B}$ is *not* topologically trivial. This can be seen for example by calculating the cohomology of X. Were it trivial, the Künneth formula would imply its first Betti number is three. On the other hand, X is an open subset of $\mathbf{C}^2 - \{0\}$, from the exact sequence

$$\mathrm{H}^1(\mathbf{C}^2 - \{0\}) \to \mathrm{H}^1(\mathrm{X}) \to \mathrm{H}^2_{\mathrm{X}(0) - \{0\}}(\mathbf{C}^2 - \{0\}) \to \mathrm{H}^2(\mathbf{C}^2 - \{0\})$$

and the Thom isomorphism theorem implies that $\dim \mathrm{H}^1(\mathrm{X}) = 1$, a contradiction.

The first de Rham cohomology group $\mathrm{H}^1(\mathrm{X}(t))$, as t varies, form a local system on B of rank 2. In view of the previous item, this local system is not trivial.

1.3. Local system and fundamental group. A topological space $\pi \colon \mathrm{E} \to \mathrm{X}$ over X is said to be an *espace étalé* over X if π is a local homeomorphism. In general, an espace étalé over X is not a separated space over X, for instance the natural proejction from the affine line with a doubled origin to the ordinary affine line is not separated.

If $\mathscr{F}$ is a presheaf of sets on a topological space X, we can form its *associated espace étalé* $\pi \colon \mathrm{E}_{\mathscr{F}} \to \mathrm{X}$ as follows:

- the ambient set of $\mathrm{E}_{\mathscr{F}}$ is the disjoint union $\coprod_{x \in \mathrm{X}} \mathscr{F}_x$ (recall that elements in the stalk $\mathscr{F}_x$ are called *germs*);
- for each open subset U of X and $s \in \mathscr{F}(\mathrm{U})$, define $\mathrm{G}_{[\mathrm{U},s]}$ to be the collection of germs s_x when x runs through points of U.

The sets $\mathrm{G}_{[\mathrm{U},s]}$ form a basis of topology of $\mathrm{E}_{\mathscr{F}}$. With this topology the natural map $\mathrm{E}_{\mathscr{F}} \to \mathrm{X}$, which sends a germ to the point it underlies, is a local homeomorphism; for if $s \in \mathscr{F}(\mathrm{U})$, the map $x \mapsto s_x$ is a local inverse of $\pi|_{\mathrm{G}_{\mathrm{U},s}}$.

By sheaf theory, the functor

$$\mathcal{F} \mapsto \begin{bmatrix} \mathrm{E}_{\mathcal{F}} \\ \downarrow \\ \mathrm{X} \end{bmatrix}$$

is an equivalence between the category of sheaves on X and the category of *espaces étalé* over X. A quasi-inverse is given by

$$\begin{bmatrix} \mathrm{E} \\ \downarrow\pi \\ \mathrm{X} \end{bmatrix} \mapsto \Big[\mathrm{U} \mapsto \{\text{Continuous sections of } \pi \text{ over } \mathrm{U}\}\Big].$$

The following lemma is an exercise in general topology.

Lemma. *A sheaf $\mathcal{F}$ on X is locally constant if and only if $\pi\colon \mathrm{E}_{\mathcal{F}} \to \mathrm{X}$ is a covering space.*

In particular, the espace étalé of any locally constant sheaf over a simply connected topological space X is isomorphic to $\mathrm{F} \times \mathrm{X}$ for some discrete space F. Consequently the sheaf is the constant sheaf associated with F.

Now suppose X admits a simply connected covering space $u\colon \widetilde{\mathrm{X}} \to \mathrm{X}$. Let F be a right $\mathrm{G} = \mathrm{Aut}_{\mathrm{X}}(\widetilde{\mathrm{X}})$-set. Then the quotient space of the equivalence relation

$$(\phi, g(x)) \sim (\phi \cdot g, x), \forall g \in \mathrm{G},$$

denoted by $\mathrm{F} \times^{\mathrm{G}} \widetilde{\mathrm{X}}$, is still naturally a covering space over X; its sheaf of continuous sections is then a locally constant sheaf valued in F. Moreover, every locally constant sheaf on X valued in F comes in this fashion. If $\mathrm{F} = \mathbf{C}^n$, and G acts on F by linear transformations, then the resulting locally constant sheaf is a local system of rank n; conversely any local system on X is from some right G-set F.

When X is a pointed *espace délaçable* (i.e., a path connected, locally path connected, semi-locally simply connected topological space, see Bourbaki), $\mathrm{G}^{\mathrm{op}} \simeq \pi_1(\mathrm{X}, x)$. The above discussion gives an equivalence between the category of local systems and the category of finite dimensional linear representations of $\pi_1(\mathrm{X}, x)$.

A more comprehensive introduction can be found in Bourbaki.

2. Integrable connections

2.1. The definitions of an integrable connection. Let X be a complex manifold. Let M be a finite locally free $\mathcal{O}_{\mathrm{X}}$-module. As Deligne said in his book (free English translation):

> ... the ancients would define a (holomorphic) connection on M as the data which assigns to each pair of "infinitesimally close points" (x, y), an isomorphism $\gamma_{y,x} : \mathrm{M}_x \to \mathrm{M}_y$, such that this isomorphism depends in a holomorphic fashion on (x, y) and satisfies $\gamma_{x,x} = \mathrm{Id}$.
>
> Interpreted suitably, this "definition" coincides with the definition given below. But the meaning of "points" should be tweaked: it should mean "points valued in an analytic space" ...

A less intuitive definition, due to J. L. Koszul, using "covariant derivatives", is as follows. A *connection* on M is a **C**-linear map

$$\nabla\colon \mathrm{M} \to \Omega^1_{\mathrm{X}} \otimes_{\mathscr{O}_{\mathrm{X}}} \mathrm{M},$$

such that for any (locally defined) holomorphic function h and any section s of M, we have

$$\nabla(h \cdot s) = \mathrm{d}h \otimes s + h \cdot \nabla(s).$$

If ξ is a locally defined tangent vector on M, the notation $\nabla_\xi(s)$ signifies the section $\langle \xi, \nabla(s)\rangle$ of M, where $\langle \cdot, \cdot \rangle$ is the pairing between the tangent and cotangent bundle of X.

Let M and N be two finite locally free $\mathscr{O}_{\mathrm{X}}$-modules equipped with connections ∇^{M} and ∇^{N}, respectively. An $\mathscr{O}_{\mathrm{X}}$-linear morphism $\varphi\colon \mathrm{M} \to \mathrm{N}$ is said to be compatible with connections, or horizontal, if

$$\nabla^{\mathrm{N}} \circ \varphi = (\mathrm{Id} \otimes \varphi) \circ \nabla^{\mathrm{M}}.$$

A connection ∇ is *integrable* if it satisfies

$$\nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta],$$

where $[\cdot,\cdot]$ is the Lie bracket of operators. Equivalently, ∇ is integrable if $\nabla^1 \circ \nabla = 0$, where $\nabla^1 : \Omega^1_{\mathrm{X}} \otimes_{\mathscr{O}_{\mathrm{X}}} \mathrm{M} \to \Omega^2_{\mathrm{X}} \otimes_{\mathscr{O}_{\mathrm{X}}} \mathrm{M}$ is the natural extension of ∇ by "forcing the Leibniz rule".

A section s of M is said to be *horizontal* if $\nabla(s) = 0$.

Sentences like "(M, ∇) is a finite locally free $\mathscr{O}_{\mathrm{X}}$-module equipped with an integrable connection" are too long. We shall simply say "M is an integrable connection" instead.

2.2. Examples of integrable connections.

1) The free $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{O}_{\mathrm{X}}^{\oplus r}$, equipped with the exterior differential d, is an integrable connection.

Conversely, any *integrable* connection is locally trivial:

Theorem. *Let* M *be an integrable connection on* X*. Then* X *admits an open covering* $(\mathrm{U}_i)_{i\in \mathrm{I}}$*, such that* $\mathrm{M}|_{\mathrm{U}_i}$ *is isomorphic to a finite direct sum of* $(\mathscr{O}_{\mathrm{U}_i}, \mathrm{d})$*.*

The theorem can be shown by using a complex analytic version of the Frobenius theorem (see Voisin's book); alternatively it could be proved by reducing to dimension one (at the cost of introducing fancier concepts). We shall explain the second proof in §3.

The theorem has the following corollary.

Corollary. *Let* M *be an integrable connection on a complex manifold. Then the sheaf* $\mathrm{U} \mapsto \mathrm{Ker}(\nabla\colon \mathrm{M}(\mathrm{U}) \to \mathrm{M}(\mathrm{U}))$ *is a local system.*

2) If $\varphi\colon \mathrm{M} \to \mathrm{N}$ is a horizontal morphism between integrable connections, then $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ are integrable connections.

This assertion is non-trivial: the kernel and cokernel of a plain $\mathscr{O}_{\mathrm{X}}$-linear morphisms between finite locally free sheaves may not by locally free, unless its rank is locally constant.

To prove the assertion, let us verify φ has locally constant rank. The problem being local, we can prove it with X replaced by any of its open subset, which enables us to use the theorem mentioned in Example 1. For a morphism $\varphi\colon \mathcal{O}^n \to \mathcal{O}^m$ to be horizontal, it is necessary and sufficient that it is represented by a matrix with locally constant coefficients, when has locally constant rank.

In the statement, the integrability is irrelevant, since to prove the constancy of the rank, one can always restrict to curves, over which the connections are automatically integrable.

3) Let M_1 and M_2 be two finite locally free $\mathcal{O}_{\mathrm{X}}$-modules equipped with connections ∇_1 and ∇_2 respectively. Then we can define connections on the sheaves $\mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(\mathrm{M}_1, \mathrm{M}_2)$, $\mathrm{M}_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M}_2$ as follows: tensor connection formula is

$$\nabla_\xi(v_1 \otimes v_2) = \nabla_{1,\xi}(v_1) \otimes v_2 + v_1 \otimes \nabla_{2,\xi}(v_2);$$

the connection on the Hom sheaf is

$$(\nabla_w f)(v_1) = \nabla_{2,w}(f(v_1)) - f(\nabla_{1,w} v_1).$$

If M_1 and M_2 are integrable connections, $\mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(\mathrm{M}_1, \mathrm{M}_2)$ and $\mathrm{M}_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M}_2$ are integrable as well. The connection on $\mathcal{H}om_{\mathcal{O}}(\mathrm{M}, \mathcal{O})$ is called the *dual connection*, and is denoted by $\mathrm{M}^\vee$.

4) Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of complex manifolds. The *inverse image* $f^*\mathrm{M}$ of an integrable connection M on Y is again an integrable connection on X.

5) Let E be a local system on X. Define $\mathrm{M} = \mathrm{E} \otimes_{\mathbf{C}_{\mathrm{X}}} \mathcal{O}_{\mathrm{X}}$. Let U be an open subset of X on which E is trivial, and let $\boldsymbol{v} = (v_1, \dots, v_r)$ is a basis of E. Then a section of M over U is given by $\sum_{i=1}^r f_i v_i$, where $f_i \in \mathcal{O}_{\mathrm{X}}(\mathrm{U})$. Define

$$\nabla \sum_{i=1}^r f_i v_i = \sum_{i=1}^r \mathrm{d} f_i \otimes v_i.$$

As the transition functions between the various frames $\boldsymbol{v}$ are locally constant, it follows that ∇ globalizes, thereby giving rise to an integrable connection on M.

2.3. The "trivial" Riemann–Hilbert correspondence. In view of the theorem stated in Example 2.2/1 and Example 2.2/5, we find that the functors

$$[\mathrm{M},\ \text{an integrable connection}] \mapsto \operatorname{Ker} \nabla,$$

and

$$[\mathrm{E},\ \text{a local system}] \mapsto (\mathrm{E} \otimes_{\mathbf{C}} \mathcal{O}_{\mathrm{X}}, \nabla)$$

are equivalences between the category of integrable connections on X and the category of local systems on X.

3. Proof of Theorem 2.2/1

In this section, we present Deligne's proof of Theorem 2.2/1. In fact, he proved a more general theorem which is valid in the *relative setting*, i.e., a theorem about a *smooth morphism* $f\colon \mathrm{X} \to \mathrm{S}$ between complex manifolds (or complex analytic spaces). When S reduces to a point, we thus get an *absolute* theorem, i.e., a theorem about the complex manifold X.

Stating the relative version of the theorem allows us to prove the theorem by *dévissage*: "unscrewing" the proof into small steps which are not difficult. Formalism aside, the core is still Cauchy's theorem (with parameters) in dimension one.

3.1. Definitions.

1) A morphism $f\colon \mathrm{X} \to \mathrm{S}$ between complex manifolds is called a *smooth morphism*, provided each point x of X possesses a neighborhood U, such that there exists commutative diagram

$$\begin{array}{ccc} \mathrm{U} & \overset{j}{\hookrightarrow} & \mathbf{C}^n \times \mathrm{S} \\ & {\scriptstyle f|_{\mathrm{U}}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & \mathrm{S} & \end{array}$$

in which U is an open immersion.

2) Let $f\colon \mathrm{X} \to \mathrm{S}$ be a smooth morphism of complex manifolds. Then the relative cotangent bundle $\Omega^1_{\mathrm{X/S}}$ is defined by the exact sequence

$$0 \to f^*\Omega^1_{\mathrm{S}} \to \Omega^1_{\mathrm{X}} \to \Omega^1_{\mathrm{X/S}} \to 0.$$

Note that the left arrow is injective, by virtue of smoothness of f.

3) Let M be a coherent $\mathcal{O}_{\mathrm{X}}$-module. A *relative connection* on M is an $f^{-1}\mathcal{O}_{\mathrm{S}}$-linear map

$$\nabla\colon \mathrm{M} \to \Omega^1_{\mathrm{X/S}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M}.$$

satisfying the "Leibniz rule": $\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s$; except here we use $\mathrm{d}f$ to mean the image of usual $\mathrm{d}f$ in $\Omega^1_{\mathrm{X/S}}$. It is said to be *integrable* if $\nabla \circ \nabla = 0$.

4) A *relative local system* on X is a sheaf of $f^{-1}\mathcal{O}_{\mathrm{S}}$-modules, locally isomorphic to $f^{-1}\mathcal{F}$ for a coherent sheaf $\mathcal{F}$ on S.

Theorem. *Let* $f\colon \mathrm{X} \to \mathrm{S}$ *be a smooth morphism between complex manifolds. Let* (M, ∇) *be a relative integrable connection on* X*. Then* $\mathrm{U} \mapsto \operatorname{Ker} \nabla$ *is a relative local system on* X*.*

3.2. Initial case: $\mathrm{X} = \mathrm{S} \times \Delta$, $\mathrm{S} = \Delta^n$, $f = \mathrm{pr}_1$, M free.

Let s_0 be the zero section $\mathrm{S} \simeq \mathrm{S} \times \{0\} \hookrightarrow \mathrm{X}$. Then $\mathcal{F} = s_0^*\mathrm{M}$ is a free $\mathcal{O}_{\mathrm{S}}$-module on S. For each section v of $\mathcal{F}$, the existence and uniqueness theorem of Cauchy problem with parameters ensures that there is a horizontal section $\tilde{v}$ of M such that $v = \tilde{v}|_{\mathrm{S}\times\{0\}}$. Let $\boldsymbol{v}$ be a frame of $\mathcal{F}$. Apply the above to $\boldsymbol{v}$ yields a frame of $\tilde{\boldsymbol{v}}$ of M which is horizontal.

3.3. Second case: $\mathrm{X} = \mathrm{S} \times \Delta$, $\mathrm{S} = \Delta^n$, $f = \mathrm{pr}_1$.

By shrinking S and X to smaller disks, we can assume M admits a presentation

$$\mathrm{M}_1 \overset{d}{\to} \mathrm{M}_0 \overset{\epsilon}{\to} \mathrm{M} \to 0$$

where M_0 and M_1 are free. By restricting further, we could assume M_1 and M_0 are equipped with connections such that ϵ and d are compatible with the connection. (If $\boldsymbol{e}$ is a basis of M_0, it suffices to determine $\nabla_0(\boldsymbol{e})$. One can define it to be any one in the preimage of $\nabla(\epsilon\boldsymbol{e})$; define ∇_1 similarly.)

Since the relative dimension is one, ∇_0, ∇_1 are automatically integrable. By the initial case, there exists coherent $\mathcal{O}_{\mathrm{S}}$-modules $\mathcal{F}_0$ and $\mathcal{F}_1$ such that $(\mathrm{M}_i, \nabla_i) \simeq \mathcal{O}_{\mathrm{X}} \otimes_{f^{-1}\mathcal{O}_{\mathrm{S}}} \mathcal{F}_i$. It follows that $\mathrm{M} \simeq \mathcal{O}_{\mathrm{X}} \otimes_{f^{-1}\mathcal{O}_{\mathrm{S}}} \mathcal{F}_0/d\mathcal{F}_1$.

3.4. Third case: f is of relative dimension 1. This follows immediately from the second case as S is locally isomorphic to a polydisk. We remark that this also works if S is singular, i.e., an arbitrary complex analytic space. This is because locally S can be embedded into a polydisk Δ^n and coherent sheaves on S are locally restricted from Δ^n.

3.5. General case. Proceed by induction on the relative dimension n. The base case ($n = 0$) is trivial, we assume $n > 0$.

The problem being local, we can assume $X \cong S \times D^n = S \times D^{n-1} \times D$. Set $X_0 = S \times D^{n-1} \times \{0\}$, and $p\colon X \to X_0$ be the projection to the first two factors.

The restriction of M to X_0 is a relative integrable connection M_0 on X_0. By inductive hypothesis,

$$M_0 \cong \mathcal{O}_{X_0} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_S} \mathcal{F} \tag{I}$$

for some coherent sheaf $\mathcal{F}$ on S.

On the other hand, (M, ∇) also defines an integrable connection relative to $S \times D^{n-1} \simeq X_0$. Thus there exists a coherent sheaf $\mathcal{G}$ on X_0 such that

$$M \cong \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_{X_0}} \mathcal{G}. \tag{II}$$

As we have seen in the first and second case, $\mathcal{G}$ is isomorphic to M_0. Combining (I) and (II) gives an isomorphism $\alpha\colon M \xrightarrow{\sim} \mathcal{O}_X \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_S} \mathcal{F}$. It satisfies:

- the restriction of α to X_0 is horizontal;
- α is "relatively horizontal" for p.

From these two points we can deduce the horizontality of α, as follows.

Let v be a section of $\mathcal{F}$. Then v can be viewed as a horizontal section of M_0. The second point implies that $\nabla_{\partial_n} v = 0$. It follows from the integrability that

$$\nabla_{\partial_n} \nabla_{\partial_i} v = \nabla_{\partial_i} \nabla_{\partial_n} v = 0.$$

Thus $\nabla_{\partial_i} v$ are horizontal relative to p. By the first point, however, $\nabla_{\partial_i} v = 0$ on X_0. Therefore $\nabla v = 0$, meaning v is indeed horizontal for the relative connection M. This proves that α is horizontal and finishes the proof of Theorem 3.1.