A THEOREM OF CAUCHY

Reference. Otto Forster. *Lectures on Riemann surfaces* (1981). §11.

Fix a positive real number R (could be $+\infty$). Let $\Delta = \{t \in \mathbf{C} : |t| < \mathrm{R}\}$. Let S be a complex manifold (or a complex analytic space). Let $\mathrm{A}(s,t)$ be an $(n \times n)$-matrix whose entries are holomorphic functions on $\mathrm{S} \times \Delta$.

Consider the following family of systems of ordinary differential equations parametrized by $s \in \mathrm{S}$:

$$(*) \qquad \frac{\partial \boldsymbol{x}}{\partial t} - \mathrm{A}(s,t)\boldsymbol{x}(s,t) = 0, \quad \boldsymbol{x}(s,t) \in \mathcal{O}(\mathrm{S} \times \Delta)^{\oplus r}.$$

Theorem. *Fix $\boldsymbol{x}_0 \in \mathcal{O}(\mathrm{S} \times \{0\})^n$. There exists a unique $\boldsymbol{x} \in \mathcal{O}(\mathrm{S} \times \Delta)^{\oplus n}$ satisfying $(*)$ and $\boldsymbol{x}(s,0) = \boldsymbol{x}_0(s)$.*

Uniqueness. Write

$$\mathrm{A}(t) = \sum_{\nu=0}^{\infty} \mathrm{A}_\nu(s) t^\nu, \quad \mathrm{A}_\nu \in \mathrm{M}_n(\mathcal{O}(\mathrm{S})).$$

Let $\boldsymbol{x}(t) = \sum_{\nu=0}^{\infty} c_\nu(s) t^\nu$ be a solution of $(*)$. Then

$$\sum_{k=1}^{\infty} k c_k t^{k-1} = \sum_{k=0}^{\infty} \left(\sum_{\mu+\nu=k} \mathrm{A}_\mu c_\nu \right) t^k.$$

Thus

$$c_{k+1} = \frac{1}{k+1} \sum_{\nu=0}^{k} \mathrm{A}_{k-\nu} c_\nu, \quad k = 0, 1, \dots .$$

Once $c_0(s) = \boldsymbol{x}(s,0)$ is determined, the above recursive equation determines all $c_k(s)$, $k \in \mathbf{N}$. This proves the uniqueness.

Existence. Keep the above notation above. Fix any compact polydisk P of S, and a real number $0 < r < \mathrm{R}$.

Since the entries of $\mathrm{A}(s,t)$ are analytic on $\mathrm{P} \times \{|t| < r\}$, there exists $\mathrm{N} \in \mathbf{N}$ such that

$$\|\mathrm{A}_\nu(s)\|_{\mathrm{P},\infty} = \max\{|a_{ij,\nu}(s)| : 1 \le i, j \le n; s \in \mathrm{P}\} \le \mathrm{N} r^{-\nu-1},$$

where $a_{ij,\nu}(s)$ are the entries of A_ν.

Define

$$b_{ij}(t) = \frac{\mathrm{N}}{r}\left(1 - \frac{t}{r}\right) = \frac{\mathrm{N}}{r} \sum_{\nu=0}^{\infty} \frac{t^\nu}{r^\nu}.$$

Consider an auxiliary equation (independent of s)

$$\boldsymbol{y}' = \mathrm{B}(t)\boldsymbol{y},$$

with initial condition $\boldsymbol{y}(0) = (\mathrm{K}, \dots, \mathrm{K})^{\mathsf{T}}$. Here, K is any real number which is larger than $\|\boldsymbol{x}(s)_0\|_{\mathrm{P},\infty} < \infty$.

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The auxiliary equation is easily solved by $\boldsymbol{y}(t) = (\psi(t), \dots, \psi(t))^{\mathsf{T}}$, where

$$\psi(t) = \mathrm{K}\left(1 - \frac{t}{r}\right)^{-n\mathrm{N}}.$$

The solution $\boldsymbol{y}(t)$ is visibly convergent on $\{|t| < r\}$.

If we write

$$\boldsymbol{y}(t) = \sum_{\nu=0}^{\infty} \gamma_\nu t^\nu,$$

then by the uniqueness step the vectors γ_ν and the matrices B_ν satisfy the recursive relation

$$\gamma_{k+1} = \frac{1}{k+1} \sum_{\nu=0}^{k} \mathrm{B}_{k-\nu}\gamma_\nu.$$

By induction and the hypothesis on the matrix B, we conclude that $\boldsymbol{y}(t)$ is a majorant of the series $\boldsymbol{x}(s,t)$ on $\mathrm{P} \times \{|t| < r\}$, i.e.,

$$\|c_k(s)\|_{\mathrm{P},\infty} < \gamma_k.$$

This forces $\boldsymbol{x}$ to be convergent in $\mathrm{P} \times \{|t| < r\}$. Since r is arbitrary, $\boldsymbol{x}(t)$ converges on $\mathrm{P} \times \Delta$. Since P is arbitrary, the uniqueness implies $\boldsymbol{x}$ exists globally on $\mathrm{S} \times \Delta$.