Degeneration of slopes

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Introduction

This article proves results on "degeneration of coniveau" for varieties over fields of characteristic p.

When the ambient field is a finite field, our result can be roughly described as follows: according to a theorem of Esnault [11], there is a simple, cohomological condition (the "coniveau $\geq$ 1" condition) on proper varieties, which, by the Lefschetz fixed point formula, can force the existence of points over the finite field; when a family of varieties with such a condition degenerates, the cohomological condition retains, so the degenerate member should still contain a rational point over the finite field.

In [10], Hélène Esnault proved a theorem on the existence of points for a singular specailization of a smooth scheme X over a local field that are of geometric coniveau ≥ 1:

Theorem 0.0.1 (Esnault [10])**.** *Let $\mathcal{X}$ is a regular scheme over the ring of integers of a local field K whose residue field has q elements, such that $\mathcal{X}_K$ has geometric coniveau ≥ 1. Then the number of rational points of the reduction of $\mathcal{X}$ is congruent to 1 modulo q.*

Esnault asked the question whether the geometric coniveau condition could be replaced by certain cohomological coniveau conditions.

In the mixed characteristic situation this is answered affirmatively by the work [2] of Berthelot, Esnault and Rülling, where the Hodge coniveau condition is used instead of the geometric one.

Theorem 0.0.2 (Berthelot, Esnault, Rülling [2])**.** *Let K be a local field of characteristic 0 whose residue field has q elements. Let $\mathcal{X}$ be a regular scheme over the ring of integers over K. Suppose $\mathcal{X}_K$ has Hodge coniveau ≥ 1, then the number of rational points of the reduction of $\mathcal{X}$ is congruent to 1 modulo q.*

The main result of this article is that in the equal characteristic case, the existence of points of the singular specialization can be derived under the slope coniveau ≥ 1 condition on the "general fibers" of the family.

Our first result, stated in terms of étale cohomology, is the following.

Theorem 0.0.3. *Fix an integer $i > 0$ and a real number $0 \leq s \leq 1$. Let $f : X \to B$ be a proper, generically smooth, morphism of nonsingular varieties over a finite field k of q elements. Assume that there exists a nonempty Zariski open subset U of B such that for all closed points t in U, the slopes of $\mathrm{H}^i(X_{\bar{t}}, \mathbb{Q}_\ell)$ are $\geq s$. Then the same is true for all closed points of B_0.*

In particular, if $s = 1$, then the number of k-points of the singular fiber is congruent to 1 modulo q.

The proof of the theorem uses a method found in SGA 7.

The rest of this article prove analogue results for rigid cohomologies of varieies defined over an arbitrary perfect field of characteristic p. We have two results, one is conditional, relying on the existence of Leray spectral sequence in rigid cohomology; the other is unconditional, but assumes the degeneration is semistable.

Proposition 0.0.4. *Let $f : X \to B$ be a proper, generally smooth morphism of smooth projective varieties over a perfect field k of characteristic $p > 0$. Let $0 \leq s \leq 1$ be a rational number. Assume that:*

(1) *there exists a nonempty Zariski open subset U of B, such that for all closed points $t \in U$, the fiber X_t of f over t has slopes $\geq s$ in all cohomology degrees,*
(2) *the higher direct images $R^i f_{\mathrm{rig}*}\mathcal{O}_{X/K}$ are coherent on some nonempty open subset of B.*

Then for any closed point $t \in B$, the condition (1) holds.

Proposition 0.0.5. *Let $f : X \to B$ be a proper, generally smooth morphism of nonsingular, quasi-projective, varieties over a perfect field k of characteristic $p > 0$. Let $0 \leq s \leq 1$ be a rational number. Fix an integer i. Assume that:*

(1) *there exists a nonempty Zariski open subset U of B, such that for any closed point $t \in U$, the slopes of $\mathrm{H}^i_{\mathrm{rig}}(X_t)$ are $\geq s$,*
(2) *all fibers of f are semistable.*

Then for any closed point $t \in B$, the condition (1) holds.

In the last chapter of this article, we record some geometric applications of our theorems.

CHAPTER 1

Levels and slopes

In this chapter, we review the various conivieau conditions and how they relate to each other.

1.1. Geometric levels of cohomology groups

There are several coniveau conditions that one can put on different cohomology groups.

Definition 1.1.1. Let X be a smooth, projective variety defined over a field K. Let $\overline{K}$ be a chosen separable closure of K. Let H be a Weil cohomology theory for varieties over $\overline{K}$. We say X is of (geometric) *conieveau* ≥ 1 at cohomological degree i (with respect to the cohomology theory H), if there is a divisor D of $X_{\overline{K}} = X \otimes_K \overline{K}$ such that the natural map

$$\mathrm{H}^i(X_{\overline{K}}, X_{\overline{K}} \setminus D) \to \mathrm{H}^i(X_{\overline{K}})$$

is surjective. In this case, we say the divisor D *supports the cohomology of* X.

The geometric coniveau condition can be interpreted slightly differently without using the relative cohomology groups. In the following proposition we give an illustration of this using étale cohomology. The argument can be modified so that a similar result could also be obtained for the Betti theory for varieties over a field of characteristic 0.

Proposition 1.1.2. *Let X be a smooth, projective, irreducible scheme of dimension n over an algebraically closed field k. Let ℓ be a prime number different from the characteristic of k. Then X has geometric coniveau one at degree m with respect to the ℓ-adic cohomology if and only if there is a smooth, proper scheme Y of dimension $n-1$, a morphism $f : Y \to X$, such that*

$$f^* : \mathrm{H}^{2n-m}(X, \mathbb{Q}_\ell) \to \mathrm{H}^{2n-m}(Y, \mathbb{Q}_\ell)$$

is injective.

Proof. Let D be the variety that supports the coniveau of the cohomology of X. Let $f : Y \to D$ be an alteration of D by a smooth, projective k-variety Y. We shall show that Y is the desired variety.

First, we use spread-out argument to reduce the problem to the case when k is an algebraic closure of a finite field.

Since X, D and Y by defined in projective spaces by finitely many polynomial equations, the number of coefficients of these polynomial is finite. Therefore, there exist:

- a finitely generate $\mathbb{Z}$-algebra R,
- smooth, projective morphisms $\mathcal{X} \to \mathrm{Spec}(R)$, $\mathcal{Y} \to \mathrm{Spec}(R)$,
- a projective morphism $\mathcal{D} \to \mathrm{Spec}(R)$,
- a closed immersion $\mathcal{D} \to \mathcal{X}$,
- a generically finite dominant morphism $\mathcal{Y} \to \mathcal{D}$,
- a morphism $\eta : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ such that
  - $X = \mathcal{X} \otimes_\eta k$,
  - $Y = \mathcal{Y} \otimes_\eta k$,
  - $D = \mathcal{D} \otimes_\eta k$.

By the smooth base change theorem, after shrinking $\mathrm{Spec}(R)$ if possible, we know that for all closed points t of $\mathrm{Spec}(R)$, the variety $\mathcal{X} \otimes_R \kappa(t)$ has coniveau ≥ 1 and $\mathcal{D} \otimes_R \kappa(t)$ supports the cohomology. If we know the result over $\overline{\kappa(t)}$, using smooth base change theorem again deduces the result over k. Therefore, we can assume k is already an algebraic closure of a finite field.

By the hypothesis, we have a surjection $\mathrm{H}^m_D(X) \to \mathrm{H}^m(X)$. By the Verdier duality, we have an isomorphism

$$\mathrm{H}^m_D(X) \cong \mathrm{Hom}(\mathrm{H}^{2n-m}(D), \mathbb{Q}_\ell(-n)).$$

Taking dual vector spaces, one concludes that the restriction map

$$\mathrm{H}^{2n-m}(X) \to \mathrm{H}^{2n-m}(D)$$

in injective. Now let $f : Y \to D$ be an alteration of D by a smooth k-variety Y. Then the composition

$$\mathrm{H}^m(X) \to \mathrm{H}^m(D) \to \mathrm{H}^m(Y).$$

is injective, since we can take the weight m graded piece and apply Lemma 1.1.3 below and the strictness of the weight filtration. $\square$

Lemma 1.1.3. *Let $f : X \to Y$ be a morphism of proper varieties over a finite field k. Let ℓ be a prime number that is not equal to the characteristic of k. Then the induced map*

$$\mathrm{Gr}_i^W(f^*) : \mathrm{Gr}_i^W \mathrm{H}^i(Y_{\overline{k}}, \mathbb{Q}_\ell) \to \mathrm{Gr}_i^W \mathrm{H}^i(X_{\overline{k}}, \mathbb{Q}_\ell)$$

is injective.

Proof. Apply the argument of [21] Theorem 5.31, using the cubical resolution and the hypercovering spectral sequence, replacing resolution of singularity by de Jong's alteration [6]. $\square$

Another demonstration of the implications of the geometric coniveau condition is the following.

Lemma 1.1.4. *Let X be a smooth K-variety of geometric conieveau ≥ 1 at degree one with respect to the ℓ-adic cohomology theory. Then $\mathrm{H}^1(X_{\overline{K}}, \mathbb{Q}_\ell) = 0$.*

Proof. When the support of D of the cohomology is nonsingular, then this follows from the "Thom isomorphism theorem" (or the purity theorem in étale cohomology, proven in the current situation by Grothendieck himself, and in a more general setting by Ofer Gabber), which asserts that

$$\mathrm{H}^i_D(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^{i-2}(D_{\overline{K}}, \mathbb{Q}_\ell(-1)).$$

Since there étale cohomology vanishes negative degree, taking $i = 1$ proves the lemma in this case. In general, we can write D as a stratification $D = \coprod_i D_i$ into locally closed subschemes such that each D_i is a nonsingular. One proves by induction that removing a codimension ≥ 2 nonsingular closed subscheme won't change the H^2. It follows that we can remove all the codimension two or higher strata in the decomposition without changing the second cohomology of X. But the remaining part is a nonsingular divisor inside a nonsingular variety, so we win. $\square$

1.2. Hodge coniveau

Definition 1.2.1. Let K be a field of characteristic zero. A smooth, proper scheme of finite type over k is said to have *Hodge coniveau* ≥ 1, if $\mathrm{H}^i(X, \mathcal{O}_K) = 0$, for all $i > 0$.

Lemma 1.2.2. *Let X be a smooth, proper variety over a field K that can be embedded into $\mathbb{C}$. Suppose X has geometric conieveau one at degree i with respect to the Betti theory. Then*

$$\mathrm{H}^i(X, \mathcal{O}_X) = 0$$

That is, X has Hodge coniveau one at degree i.

Proof. Let $\sigma : K \hookrightarrow \mathbb{C}$ be the embedding of K into the field of complex numbers, so that X is of geometric coniveau ≥ 1 with respect to the σ-Betti theory.

Let D be the divisor that supports the coniveau. Let $f : X' \to X$ be a strong resolution of singularities such that the preimage of D in X' is a simple normal crossing divisor. By the Poincaré duality, since a subvariety has nontrivial cohomology class, f induces an injection

$$f^* : \mathrm{H}^*(X) \hookrightarrow \mathrm{H}^*(X').$$

We are reduced to the case that D is a simple normal crossing divisor.

For the ease of exposition we assume that D has two irreducible components, D_1 and D_2. There is a spectral sequence whose E_1-stage reads

$$\begin{array}{c|c|c|}
\vdots & \vdots & \vdots \\
\mathrm{H}^0(D_1 \cap D_2, \mathbb{Q}(-2)) & \mathrm{H}^2(D_1, \mathbb{Q}(-1)) \oplus \mathrm{H}^2(D_1, \mathbb{Q}(-1)) & \mathrm{H}^4(X, \mathbb{Q}) \\
0 & \mathrm{H}^1(D_1, \mathbb{Q}(-1)) \oplus \mathrm{H}^1(D_1, \mathbb{Q}(-1)) & \mathrm{H}^3(X, \mathbb{Q}) \\
0 & \mathrm{H}^0(D_1, \mathbb{Q}(-1)) \oplus \mathrm{H}^0(D_1, \mathbb{Q}(-1)) & \mathrm{H}^2(X, \mathbb{Q}) \\
0 & 0 & \mathrm{H}^1(X, \mathbb{Q}) \\
0 & 0 & \mathrm{H}^0(X, \mathbb{Q}) \\
\hline
\end{array}$$

whose horizontal differentials are the Gysin push forwards, and abuts to the cohomology of $X \setminus D$. Deligne [8] had proven this spectral sequence degenerates at E_2 (morally speaking, starting at E_2, all the differentials mess up the weights, hence are zero). Since X has geometric conieveau one, the restriction maps $\mathrm{H}^i(X) \to \mathrm{H}^i(X \setminus D)$ are zero. It follows that there is nothing in $\mathrm{H}^i(X)$ that survives in E_2, as the right most column of the spectral sequence contributes a factor of the cohomology of $X \setminus D$. The results follows from the Hodge decomposition of the divisors. □

Grothendieck formulated his generalized Hodge conjecture in the 1960s. The conjecture asserts that the other side of the implication should also hold: namely, Grothendieck conjectures that a smooth projective variety that has Hodge coniveau ≥ 1 should also be of geometric coniveau ≥ 1 with respect to any σ-Betti theory.

What are the implications of the Hodge coniveau ≥ 1 condition in a family?

Being of Hodge coniveau one passes to any smooth fiber in a family, since for smooth proper families over a field of characteristic zero, the Hodge numbers are constant upon variation. Now let us consider what happened when the family acquires singularities.

1.2.3. Consider a proper flat morphism of smooth algebraic varieties $f : X \to B$ of relative dimension n. Let $0 \leq i \leq n$. Assume that there exists a smooth fiber X_t such that

$$\mathrm{H}^i(X_t, \mathcal{O}_{X_t}) = 0. \tag{1.2.4}$$

That is, assume the fibers of f are of Hodge coniveau one at degree i.

Esnault, [9], Theorem 1.1, proved that if we assume (1.2.4) for *all* $i > 0$ then one can conclude even for singular fibers, one still has (1.2.4), valid for *all* $i > 0$.

Her result did not separate the various indices i. It is hoped that some Hodge theoretic input would separate the indices in question. This is indeed the case.

Proposition 1.2.5. *Let notations be as in Situation 1.2.3. Let* $i > 1$ *be an integer. Suppose* (1.2.4) *holds for one smooth fiber of* f*. Then it holds for all fibers of* f *(with reduced scheme structures).*

Remark 1.2.6. It suffices to prove the result under the hypothesis that B is a smooth curve, since an two points on B are contained in a complete intersection curve. By Bertini's theorem the pullback of X over the curve will remain smooth over $\mathbb{C}$. Henceforth we will assume $\dim B = 1$.

Proof of Proposition 1.2.5. Without loss of generality, assume $b \in B$ is the only point with singular fiber. Let $j : U \to B$ be the open immersion of the complement of b in B By the decomposition theorem [1], we have a decomposition

$$Rf_*\mathbb{C}_X = \bigoplus_i j_*(R^m f_*\mathbb{C})[-i] \oplus T_i[-i] \tag{1.2.7}$$

in the derived category of B, where T_i are some sheaves of finite dimensional $\mathbb{C}$ vector spaces supported solely on b. From the decomposition (1.2.7), we know that the ith cohomology of the fiber X_b, computed by restricting the left hand side to X_b, then taking ith cohomology, is equal to the fiber $j_*(R^i f_*\mathbb{C})_b \oplus T_i$. It turns out both summands have naturally defined Hodge filtrations.

The vector space $j_*(R^i f_*\mathbb{C})_b$ constitutes the local monodromy invariants of the cohomology of the nearby smooth fibers. By the work of Schmid [23] it has a natural limit Hodge filtration, which is part of the limit mixed Hodge structure he puts on a degenerating family. By definition, the limit Hodge filtration is the limit

$$F^\bullet = \lim_{\mathrm{Im}(z)\to\infty} \exp(-zN) \cdot F^\bullet(z), \tag{1.2.8}$$

where N is the logarithm of the unipotent part of the local monodromy operator, z is the coordinate of the upper half plane. Schmid proves (loc. cit. "nilpotent orbit theorem") that the limit exists. From (1.2.8) it follows that if the nearby Hodge filtrations satisfies $\mathrm{gr}^0_F = 0$, then the limit Hodge filtration also satisfies $\mathrm{gr}^0_F = 0$.

By the usual Wang sequence, we know that T_i is the local monodromy coinvariant of H^{i-1} of the nearby fibers, twisted by $\mathbb{C}(-1)$. Because of the Tate twist, its Hodge filtration also has zero associated graded at degree 0.

Putting together the above information, we see the Hodge filtration of $\mathrm{H}^i(X_b)$ satisfies $\mathrm{gr}^0_F = 0$, since we know the natural Hodge filtration of X_b put by Deligne *is compatible with the natural Hodge filtrations of the above topological components*. □

Remark 1.2.9. As can be read off from the above argument, the proof of the proposition has two pieces: one is the use the invariant cycle theorem, which relates the cohomology of the general fiber to the special one. The second is Schmid's limit formula (1.2.8), which tells how to pass from a general fiber to a "nearby fiber".

The local invariant cycle theorem is available in the ℓ-adic setting, as shown by Deligne in [7]. The limit formula is no longer available when we switch to ℓ-adic cohomology.

1.3. Rigid cohomology and slopes

First we fix some notations.

1.3.1. Let p be a prime number. Let k be a perfect field of characteristic p. Let $\sigma : k \to k$ be the Frobenius automorphism of k. Let $W(k)$ be the ring of Witt vectors of k. Let $K = W(k)[1/p]$ be the fraction field of $W(k)$.

1.3.2. Rigid cohomology for proper varieties. Let X be a smooth, proper, connected variety over k. Then the rigid cohomology of X is a Weil cohomology theory. At this point, let us recall the definition of the rigid cohomology for a proper variety, which will be sufficient for our purpose.

For simplicity, we also assume that X can be embedded into a smooth, proper formal scheme $\mathfrak{P}$ over $W = W(k)$. This is satisfied for example if X is projective.

Let P be the rigid generic fiber of the formal scheme $\mathfrak{P}$, defined by Raynaud. Then there is a specialization morphism of ringed topoi:

$$\mathrm{sp} : (P, \mathcal{O}_P) \to (\mathfrak{P}, \mathcal{O}_{\mathfrak{P}}).$$

The preimage $]X[_P = \mathrm{sp}^{-1} X$ is an admissible rigid analytic subspace of P, called the *tube* of X in P. The *convergent cohomology* of X is defined to be the hypercohomology of the de Rham complex

$$\mathrm{H}^\bullet_{\mathrm{rig}}(X/K) = \mathrm{H}^\bullet(R\Gamma(]X[_P, \Omega^\bullet_{]X[_P})).$$

Since we have assumed X is proper, the convergent cohomology defined above is just the rigid cohomology of X.

Although the Frobenius morphism of X does not lift to the rigid analytic space $]X[_P$, miraculously it does act on the rigid cohomology in a semi-linear way. If $\sigma : W \to W$ is the canonical Frobenius lift on the Witt vectors, and by abuse of notations $\sigma : K \to K$ its extension to K, then for any X there is also a Frobenius action $\varphi : \mathrm{H}^\bullet_{\mathrm{rig}}(X/K) \to \mathrm{H}^\bullet_{\mathrm{rig}}(X/K)$ such that for all $c \in K$, $\varphi(cv) = \sigma(c)\varphi(v)$; i.e., φ is σ-semilinear.

At this point, we recall the Dieudonné-Manin classifications of these semilinear algebraic objects.

Definition 1.3.3. Let notations be as in (1.3.1). A *φ-module* is a finite dimensional vector space D over K together with an additive endomorphism $\varphi : D \to D$, such that $\varphi(cv) = \sigma(c)v$ for all $c \in K$ and $v \in D$.

If $e = \{e_1, \ldots, e_n\}$ is a basis of D, and $\varphi(e_i) = \sum_j a_{ji} e_j$. Then the matrix $A = [\varphi]_e = (a_{ij})$ uniquely characterizes φ; the matrices of φ under two different bases is related $[\varphi]_e = \sigma(P)[\varphi]_{e'} P^{-1}$, where P is the transition matrix of two bases e and e'.

We say D is *isoclinic* (or *pure*) of slope $\mu = d/h$, where $d, h \in \mathbb{Z}$), $h \geq 1$, if there is a $W(k)$-lattice $M \subset D$ such that $p^{-d}\varphi^h M = M$.

For example, the rigid cohomology of a variety over k is a φ-module (the finite-dimensionality was a nuisance, but it now has been resolved, see e.g., [12]).

Now we can formulate the Dieudonné-Manin classification theorem of φ-modules.

Theorem 1.3.4 (Dieudonné-Manin)**.** *Let D be a φ-module. Then there is a direct sum decomposition $D = \bigoplus_{\mu \in \mathbb{Q}} D_\mu$, such that D_μ is isoclinic of slope μ.*

The rational numbers appeared in the decomposition are called the *slopes* of D.

We say a variety X has *slope coniveau* ≥ 1 if the slopes of $\mathrm{H}^i_{\mathrm{rig}}(X/K)$, when viewed as a φ-module, are all at least 1, for all $i \geq 1$ (Note: this is not a standard terminology). In the realm of rigid cohomology, a smooth variety over a finite field k with geometric coniveau ≥ 1 also has slope coniveau ≥ 1. The proof follows from a dévissage argument on the singular locus of the divisor that supports the cohomology and the "purity" result in rigid cohomology. A detailed proof, which we omit as we find it is of the same vein of many proofs that we will present below, can be found for example in Esnault's article [11].

1.3.5. The theorem of Mazur [17] and Ogus [3] relates the notions of slope coniveau and Hodge coniveau. A form of this theorem asserts that if $\mathcal{X}$ is smooth proper over W, and $\mathcal{X} \otimes_W K$ has Hodge coniveau one, then the reduction $\mathcal{X} \otimes_W k$ has slope coniveau one. The theorem is further generalized by Berthelot-Esnault-Rülling [2] to the case when $\mathcal{X}$ is allowed to have singular reduction (although the total space is assumed to be regular).

In this article we prove an "equal characteristic" analogue of their result, either under a "coherence" hypothesis or under a semistability hypothesis.

Proposition 1.3.6. *Let $f : X \to B$ be a proper, generically smooth morphism of smooth projective varieties over a perfect field k of characteristic $p > 0$. Let $0 \leq s \leq 1$ be a rational number. Assume that:*

(1) *for a general closed point $t \in B$, the fiber X_t of f over t has slopes $\geq s$ in all cohomology degrees,*
(2) *the higher direct images $R^i f_{\mathrm{rig}*}\mathcal{O}_{X/K}$ are coherent on a dense open subset of B.*

Then for all *closed points $t \in B$, the condition (1) hold true.*

Proposition 1.3.7. *Let $f : X \to B$ be a proper, generically smooth morphism of nonsingular, quasi-projective, varieties over a perfect field k of characteristic $p > 0$. Let $0 \le s \le 1$ be a rational number. Fix an integer i. Assume that:*

(1) *for a general closed point $t \in B$, the slopes of $\mathrm{H}^i_{\mathrm{rig}}(X_t)$ are $\ge s$,*
(2) *all fibers of f are semistable*

Then for all *closed points $t \in B$, the condition (1) hold true.*

Note that although the above proposition restricts the type of singlarities of the degeneration, it does not require the mapping extends to some projective varieties.

Both propositions are proven in §3.4.

1.3.8. A conjecture. As an analogue of the generalized Hodge conjecture, it seems reasonable to conjecture that, for any smooth, projective variety, if the slopes of at cohomological degree i are all ≥ 1, then its ith cohomology should be of (geometric) coniveau one. It might not be appropriate, but let us tentatively call this conjecture the "generalized Tate conjecture". This being said, we shall expect being of slope coniveau ≥ 1 is *motivic*.

However, a variety may have slopes lying strictly between zero and one, since slopes can be rational numbers. Being of slope $> s$ for some rational number $s \in (0, 1)$ at some cohomological degree does not seem to be a "motivic" condition. So the result proves in this article does not find its "geometric" counterpart unless $s = 1$.

1.4. Computing slopes by reductions to finite fields

So far I have not given any example of algebraic varieties whose slopes are computed. One methods of calculations is *reduction to finite fields*. This is the method we use. This method works well for smooth proper varieties, but less so for singular or open ones.

Let X be a smooth proper variety over $\mathbb{F}_q$ of pure dimension n, where q is a power of p. The information of the cohomology of X is captured by the "zeta function" of X:

$$Z_X(t) = \sum_{n=1}^{\infty} \frac{\#|X(\mathbb{F}_{q^n})|}{n} t^n.$$

The Weil conjectures, proven by Dwork, Artin-Grothendieck, P. Deligne, and others asserts that $Z_X(t)$ is of the form

$$Z_X(t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

where $P_i(t)$ are integral polynomials with $P_i(0) = 1$, such that all the complex conjugates of the reciprocal roots are algebraic integers of absolute value $q^{i/2}$ ("Weil numbers of weight i").

On the other hand, it also makes sense to talk about the p-adic absolute values of the reciprocal roots of P_i. It turns out the valuations of these reciprocal roots are precisely the *slopes* for the ith rigid cohomology of X. This builds a bridge to transport information between rigid cohomology and étale cohomology of X. Since the method of étale cohomology is more mature than the rigid cohomology (e.g., there is a good formalism of derived categories, etc.), it is more tempting to prove results in the étale side and then deduce the corresponding result for rigid cohomology. The analogue of the Propositions mentioned in the previous section in the étale realm is the following.

Theorem 1.4.1. *Fix an integer $i > 0$ and a real number $0 \le s \le 1$. Let $f : X \to B$ be a proper, generically smooth morphism of nonsingular varieties over a finite field k of q elements. Assume that for a general closed point $t \in B_0$, the slopes of $\mathrm{H}^i(X_{\bar{t}}, \mathbb{Q}_\ell)$ are $\ge s$. Then the same is true for all closed points of B_0.*

In particular, if $s = 1$, we conclude from the Grothendieck trace formula that the number of k-points of the singular fiber is congruent to 1 modulo q.

The idea of the proof of 1.4.1 is quite simple. In plain terms, one only needs the decomposition theorem of Bernstein-Beilinson-Deligne-Gabber plus the following observation:

Lemma 1.4.2. *If $\mathcal{L}$ is a lisse étale sheaf on a curve U, and $j : U \to C$ is an open immersion of smooth curves, the smallest slope of $j_*(\mathcal{L})$ at $C \setminus U$ will not be smaller than the smallest slope of $\mathcal{L}$. Here, "slope" is measured under an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_p$.*

Combining Lemma 1.4.2, the decomposition theorem, and some exact sequences of cohomology, the result will follow at once. See §2.

As we have said, the method we are going to use to prove the propositions formulated in the previous section is to "transport" the result in étale cohomology to rigid cohomology. The hypotheses such as the projectivity of the total space, and the slope constraints on all cohomological degrees, allow us to carry out the strategy. The coherence of the higher direct images in rigid cohomology is currently unproven; we need that to build up a Leray type spectral sequence for rigid cohomology. Knowing the existence of Leray on some sufficiently small nonempty open of B will be sufficient for the validity of the above theorem. In a few cases, we do know the coherence — e.g., when the singular fibers of f have semistable singularities. This is due to Morrow [18].

CHAPTER 2

Slopes for étale cohomology

In this short chapter, we prove Theorem 1.4.1. Most lemmata proven in this section are fairly standard applications of the theory of étale cohomology. Although the notations, the way of interpreting the objects, and the contexts are slightly different, the proofs of many of the lemmata are similar (or identical) to the reslts proven by Deligne in [$\mathrm{SGA7}_{\mathrm{II}}$], Exposé XXI, §5.

2.0.1. First we set up some conventions and notations.

(1) k will be a finite field of q elements of characteristic p,
(2) ℓ will be a prime number coprime to p.
(3) A *variety* over k is an finite type, geometrically irreducible scheme over k.
(4) We fix an isomorphism of abstract fields $\iota : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_p$, and we use the notation $|a|$ to denote the p-adic absolute value of $\iota(a)$ for $a \in \overline{\mathbb{Q}}_\ell$.
(5) For the ease of writing, for a k-variety S and any sheaf on S, we shall use the notation
$$\mathrm{H}^i(S, E)$$
to denote the étale cohomology group
$$\mathrm{H}^i(S_{\overline{k}}, E \otimes_k \overline{k}).$$
(6) We say the *slope* of an ℓ-adic number α is s if $|\alpha| = |q|^s$.

Definition 2.0.2. If E is an ℓ-adic sheaf on a variety S over k, then the *charge* of E is the infimum slope of all (geometric) Frobenius eigenvalues of E at closed points. (It could be $-\infty$.)

Definition 2.0.3. The *Newton polygon* of a $\overline{\mathbb{Q}}_\ell$-vector space V with Frobenius action is defined as follows: let $s_1 < s_2 < \cdots < s_m$ be all the slopes of F on V; with multiplicities $n_1, n_2, \ldots, n_m$; then the Newton polygon is the polygon with vertices $(0,0)$, $(n_1, s_1 n_1)$, $(n_1+n_2, s_1n_1+s_2n_2)$, …, $(n_1+\cdots+n_m, s_1n_1+\cdots+s_mn_m)$.

Let $f : X \to S$ be a family of smooth, projective varieties over k. Then the sheaf $E = R^n f_*(\mathbb{Q}_\ell)$ is lisse on S. It is a theorem of Grothendieck (stated for crystalline cohomology, but can be proven equally in the étale realm, see below) that the Newton polygon of the fiber of E goes up under specialization.

If f acquires singularities, then it does not make sense to compare the Newton polygons anymore since the dimensions of cohomology groups of the fibers can change. In this section, we prove that in any event the charge can only go up along degeneration. That is, the smallest slope of the cohomology groups at the point of degeneration won't be smaller than those of the general one. The major tool is the following lemma.

Lemma 2.0.4. *Let E be lisse étale on S, whose slope are all $\geq s$. Assume that* $\dim S = 1$, *S smooth and affine, then* $\mathrm{H}^1_c(S, E)$ *has slope $\geq s$.*

Proof. It suffices to settle the case when $s = 0$. In the general case, one can twist the sheaf by a rank Weil sheaf, and reduce to the case of $s = 0$. If $s = 0$, then by dévissage we can assume that E is irreducible. So it does not have monodromy invariants and monodromy coinvariants.

It follows from Grothendieck's trace formula that

$$\prod_{x \in |C_0|} \left(1 + \mathrm{Tr}(F_x) T^{\deg(x)} + \cdots \right) = \det(1 - T \cdot F : \mathrm{H}^1_c(C, E)).$$

A closed point $x \in |C_0|$ contributes to the right hand side if and only if $\deg(x) = 1$, i.e., $x \in C_0(\mathbb{F}_q)$. Summing up all terms that correspond to T we conclude that

$$\left|\mathrm{Tr}\big(F : \mathrm{H}^1_c(C, E)\big)\right| = \left|\textstyle\sum_{x \in C_0(\mathbb{F}_q)} \mathrm{Tr}(F_x)\right| \leq |q|^{\min s_x} \leq 1$$

The last inequality follows from the strong triangle inequality of the p-adic valuation. Now we can replace F by F^i and base change the curve to a degree i extension of k and do the same thing. This implies the coefficients of the polynomial $\det(1 - tF)$ have slopes ≥ 1. Hence the Newton polygon is entirely on first quadrant. Hence all the slopes must be ≥ 0. □

Corollary 2.0.5. *Let $j : U \to S$ be an open immersion of smooth curves over k. Let E be a lisse étale sheaf on U. Then the charge of $j_* E$ is equal to the charge of E.*

Proof. Again we can assume E is irreducible; so the monodromy invariant part of E is trivial. Using the exact sequence

$$0 \to j_!(E) \to j_*(E) \to (\text{skyscraper}) \to 0,$$

we get an exact sequence of cohomology:

$$\mathrm{H}^0(U, E) \to j_*(E)_{S\setminus U} \to \mathrm{H}^1_c(U, E).$$

By Lemma 2.0.4, we conclude that the slopes of of the stalks of $j_*(E)$ at points $S \setminus U$ are no smaller than the charge of E on U. □

Corollary 2.0.6. *Let $f : Z \to S$ be a smooth morphism of smooth varieties over k. Suppose that there is an open subscheme $V \subset S$, such that for all closed points $t \in V$, the Frobenius eigenvalues of $\mathrm{H}^i(Z_{\bar{t}}, \mathbb{Q}_\ell)$ are algebraic integers divisible by q, then for all closed points $t \in S$ the same result holds.*

Proof. Let α be a Frobenius eigenvalue of $R^i f_*\mathbb{Q}_\ell$ at a general point. Then the condition says that for any ι whatsoever, we have $|\iota\alpha| = |q|^s$ with $s \geq 1$. By Corollar 2.0.5, this implies the same holds for all Frobenius eigenvalues β for all fibers of $R^i f_*\mathbb{Q}_\ell$. Then we apply the following simple lemma, Lemma 2.0.7, to β/q. □

Lemma 2.0.7. *Let $\alpha \in \overline{\mathbb{Q}}_\ell$ be an algebraic number such that for all isomorphisms $\iota : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_p$ one has $\log_{|q|} |\alpha| \geq 0$, then α is an algebraic integer*

Proof. Let $P(T) = T^n + a_1 T^{n-1} + \cdots + a_n$ be the rational polynomial obtained as the product of $T - \beta$ where β runs through all the Galois conjugates of α. Then the hypothesis says that the all the slopes of the Newton polygon of $P(T)$ are ≥ 0. It follows that the end points of the polygon $(i, \log_{|q|}(a_i))$ must be ≥ 0. It follows that

$$a_i \in \mathbb{Q} \cap \mathcal{O}_{\overline{\mathbb{Q}}_\ell} = \mathbb{Z},$$

whence α satisfies an integral polynomial relation, we win. □

Now we are ready to prove Theorem 1.4.1. We begin by recalling the situation.

2.0.8. Let $f : X \to B$ be a proper morphism of nonsingular varieties over k that is smooth on an open subset U. Let $j : U \to X$ be the open immersion. Fix some closed point $b \in X \setminus U$ with $\kappa(x) = \mathbb{F}_q$. Let Y be the "special fiber" $f^{-1}(x) \otimes_k \overline{k}$. We assume that Y is singular.

There are some "local schemes" that are related to the special fiber Y. Let R be the strict local ring of b. Denote by η the generic point of Spec(R), and $\overline{\eta}$ a geometric generic point of Spec(R). Let X_R be the fiber product $X \times_B \mathrm{Spec}(R)$, and let X_η, resp. $X_{\overline{\eta}}$ be the generic, resp. geometric generic, fiber for X over R.

Lemma 2.0.9. *Let notations be as in Situation 2.0.8. Then the Frobenius eigenvalues of the monodromy invariant part of the cohomology group* $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ *are algebraic integers whose slopes are* $\geq s$

Proof. The local monodromy invariant part of $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ is precisely the stalk of the sheaf $j_* R^i f_* \mathbb{Q}_\ell$ at b. Note that by the hypothesis, the charge of $R^i f_* \mathbb{Q}_\ell|_U$ is at least s. By Corollary 2.0.5 for any isomorphism ι, the Frobenius eigenvalues of $j_* R^i f_* \mathbb{Q}_\ell$ are of slope $\geq s$. That they are algebraic integers follows from Lemma 2.0.7. □

By suitably using Poincaré duality and Corollary 2.0.5, one may deduce the following result, which is also proved in [10], Theorem 1.5(1).

Lemma 2.0.10. *Let notations be as in Situation 2.0.8. Then the Frobenius eigenvalues of the monodromy coinvariant part of the cohomology group* $\mathrm{H}^{i-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ *are algebraic integers.*

We also need the following well-known "purity theorem" for étale cohomology.

Lemma 2.0.11. *Let* Z *be a smooth scheme over* k*. Let* D *be a divisor in* Z*. Then the relative cohomology group* $\mathrm{H}^i_D(Z, \mathbb{Q}_\ell)$ *has Frobenius eigenvalues that are algebraic integers and are divisible by* q*.*

Proof. For the sake of completeness we give a proof of the lemma. One proceeds by choosing a stratification

$$\emptyset = D_m \subset D_{m-1} \subset \cdots \subset D_0 = D,$$

such that each D_i is closed in D_{i-1} and $D_{i-1} \setminus D_i$ is smooth over k. Then the relative cohomology sequence in étale cohomology gives an exact sequence:

$$\cdots \to \mathrm{H}^i_{D_i}(Z) \to \mathrm{H}^i_{D_{i-1}}(Z) \to \mathrm{H}^i_{D_{i-1}}(Z \setminus D_i) \to \cdots \tag{2.0.12}$$

in which we have suppressed the coefficients and all cohomology groups are understood as taken for the schemes base changed to $\overline{k}$. The sequence (2.0.12) plus an induction on m reduces the claim to the case when both Z and D are smooth over k. In this case, the "Thom isomorphism" provides the vertical identification of the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_D(Z) & \longrightarrow & \mathrm{H}^i(Z) \\ \downarrow \approx & \nearrow_{\varphi} & \\ \mathrm{H}^{i-2}(D)(-1) & & \end{array}$$

where φ is the usual Gysin pushforward. It follows that all the Frobenius eigenvalues α of the relative group $\mathrm{H}^i_D(Z)$ are algebraic integers divisible by q, since α/q are Frobenius eigenvalues of $\mathrm{H}^{i-2}(D)$ which is an algebraic integer. $\square$

By excision, one immediately deduces the following.

Corollary 2.0.13. *Let notations be as in Situation 2.0.8. Then the Frobenius eigenvalues of the group* $\mathrm{H}^i_Y(X_R, \mathbb{Q}_\ell)$ *are algebraic integers that are divisible by* q *for all* i.

Remark 2.0.14. We clarify how one gets the Frobenius action on the $\mathrm{H}^i_Y(X_R, \mathbb{Q}_\ell)$. The ring R is the maximal unramified extension of the completion of the local ring of the geometric point lying over b, so any automorphism of $\overline{k}$ admits a unique lift to R. In particular we can lift the Frobenius automorphism.

Proof of Theorem 1.4.1. Consider the analogue of Wang sequence obtained from the Leray spectral sequence for $X \to \mathrm{Spec}(R)$ and the Hochschild-Serre spectral sequence for $X_{\overline{\eta}} \to X_\eta$:

$$0 \to \mathrm{H}^{i-1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)_I(-1) \xrightarrow{\varphi} \mathrm{H}^i(X_\eta, \mathbb{Q}_\ell) \xrightarrow{\psi} \mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)^I \to 0.$$

Then this sequence is equivariant under the action of $\mathrm{Gal}(\overline{\eta}/\eta)$. By Lemma 2.0.9, the monodromy invariant part $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)^I$ has Frobenius eigenvalues that are of slopes $\geq s$. The kernel of ψ in the middle group is the Tate twist on the group of monodromy coinvariants, which also has integral Frobenius eigenvalues that are divisible by q (2.0.10). Since $s \leq 1$, the representation $\mathrm{H}^i(X_\eta, \mathbb{Q}_\ell)$ has integral Frobenius eigenvalues whose slopes are at least s.

Now consider the exact sequence associated to the pair (X, X_η):

$$\cdots \to \mathrm{H}^i_Y(X) \to \mathrm{H}^i(X) \to \mathrm{H}^i(X_\eta) \to \mathrm{H}^{i+1}_Y(X) \to \cdots \tag{2.0.15}$$

which is again Frobenius equivariant. By purity, Corollary 2.0.13, the Frobenius eigenvalues of the groups $\mathrm{H}^i_Y(X)$ are algebraic integers that are divisible by q, hence of slopes ≥ 1. By the the previous paragraph, $\mathrm{H}^i(X_\eta)$ has integral Frobenius eigenvalues whose slopes are at least s. The exactness of the sequence 2.0.15 implies that the Frobenius eigenvalues of $\mathrm{H}^i(X)$ are algebraic integers whose slopes are at least s. Now one uses the isomorphism $\mathrm{H}^i(X) \to \mathrm{H}^i(Y)$. $\square$

CHAPTER 3

Slopes for rigid cohomology

3.1. Rigid cohomology of varieties over a finite field

In this section, we will transport Theorem 1.4.1 to rigid cohomology of varieties over finite fields. See Theorem 3.1.7 below.

Due to the potential cancellation between the numerator and the denominator of the zeta function of a variety, our result does not separate various cohomology degrees anymore, except in the cohomological degree 1, see Theorem 3.1.5 below. One ingredient needed in our argument is the Leray spectral sequence for rigid cohomology. It exists if the "rigid direct image" is coherent, or if all the singular fibers are semistable and the total space has a smooth projective compactification (theorem of Morrow [18]).

The first set of inputs (nontrivial) we need is a result of Matsuda-Trihan, later improved by Lazda [15].

Remark 3.1.1 (Ogus's convergent isocrystals are overconvergent)**.** Let $f : Y \to U$ be a smooth, flat, projective morphism of irreducible, nonsingular, quasi-projective varieties over a field k of characteristic p. Assume that $\dim U = 1$, U is affine. Then S. Matsuda and F. Trihan [16] prove a conjecture of Berthelot, which says Ogus's convergent isocrystal $R^i f_* \mathcal{O}_{Y/K}$ has a unique overconvergent extension which we shall denote by $R^i f_* \mathcal{O}^\dagger_{Y/K}$.

3.1.2. *Hypothesis.* In the situation above, assume that we have a noncanonical isomorphism

$$\mathrm{H}^1_{\mathrm{rig},c}(U/K, R^{i-1} f_* \mathcal{O}^\dagger_{Y/K}) \oplus \mathrm{H}^2_{\mathrm{rig},c}(U/K, R^{i-2} f_* \mathcal{O}^\dagger_{Y/K}) \cong \mathrm{H}^i_{\mathrm{rig},c}(Y/K).$$

Remark 3.1.3. The hypothesis follows from the similar result for rigid cohomology groups without supports, and Poincaré duality with coefficients. The similar result for cohomology without support in turn follows from the existence of Leray spectral sequence.

In rigid cohomology, the existence of Leray spectral sequence has not been proved in full generality.

In [28], Theorem 3.4.1, the existence of Leray spectral sequences is proven under the coherence hypotheses. In particular, if the hypothesis of 1.3.6, (2) is satisfied, Hypothesis 3.1.2 will be satisfied.

A proof of the existence of the Leray spectral sequence in the case of semistable degeneration can be found in Morrow's paper [18], Remark 2.8.

Theorem 3.1.4 (Kedlaya). *Let E be an overconvergent F-isocrystal on a variety B over $\mathbb{F}_q$ of dimension n. Assume that the slopes of E are in the range $[r, s]$. Then $H^i_{\mathrm{rig},c}(B/K, E)$ has slopes in the range $[r + \max(0, i - n), s + \min(i, n)]$.*

Proof. See [14], Theorem 5.5.1. □

Theorem 3.1.5. *Let $f : X \to B$ be a flat, projective morphism of irreducible, nonsingular, projective varieties over k that is generically smooth over B. Assume that*

(1) $\dim B = 1$,
(2) *there is a nonempty, Zariski open subset $U \subset B$ such that for all geometric points $b \in B$, the slopes of the rigid cohomology of $\mathrm{H}^1_{\mathrm{rig}}(X_b/K)$ are at least s, where $s < 1$ is a fixed rational number.*
(3) *Hypothesis 3.1.2 holds.*

Then for all $b \in B$, $\mathrm{H}^1_{\mathrm{rig}}(X_b/K)$ has slope at least s.

Proof. By the proof of Theorem 1.4.1, we know that in this situation for all closed points $b \in B$, the slopes of $\mathrm{H}^1(X_b, \mathbb{Q}_\ell)$ are at least s.

We say a Frobenius eigenvalue of $\mathrm{H}^1(X, \mathbb{Q}_\ell)$ is *from X_U with multiplicity i* if it is a Frobenius eigenvalue of

$$\mathrm{Im}(\mathrm{H}^1_c(U, \mathbb{Q}_\ell) \to \mathrm{H}^1(X, \mathbb{Q}_\ell))$$

of algebraic multiplicity i.

We claim that these eigenvalues together with their multiplicities are independent of the choice of the Weil cohomology theory, and can be reconstructed solely from the zeta function of U.

To prove this, we first note that these eigenvalues are precisely the Frobenius eigenvalues of $\mathrm{H}^1_c(X_U, \mathbb{Q}_\ell)$ of weight 1. On the other hand, the Leray spectral sequence identifies $\mathrm{H}^1_c(X_U, \mathbb{Q}_\ell)$ with $\mathrm{H}^1_c(U, \mathbb{Q}_\ell)$. Therefore, the collection of Frobenius eigenvalues from U and their multiplicities can be identified with the reciprocal roots of the numerator of the zeta function of U. Here we have used the observation that all the Frobenius eigenvalues of $\mathrm{H}^1_c(U)$ are *motivic*: since $\mathrm{H}^2_c(U, \mathbb{Q}_\ell)$ is pure of weight 2, and the weights of $\mathrm{H}^1_c(U, \mathbb{Q}_\ell)$ are ≤ 1, the polynomials

$$\det(1 - tF|\mathrm{H}^1_c(U)) \quad \text{and} \quad \det(1 - tF|\mathrm{H}^2_c(U))$$

have no common roots. This proves the claim.

Having established the claim, we turn to the rigid cohomology side. We write down the exact sequence

$$\mathrm{H}^1_{\mathrm{rig},c}(X_U/K) \to \mathrm{H}^1_{\mathrm{rig}}(X/K) \to \mathrm{H}^1_{\mathrm{rig}}(X_S/K) \to \mathrm{H}^2_{\mathrm{rig},c}(X_U/K). \tag{3.1.6}$$

By the Weil conjecture for smooth projective varieties, we know the set of Frobenius eigenvalues of $\mathrm{H}^1(X, \mathbb{Q}_\ell)$ can be constructed by $\mathrm{H}^1_{\mathrm{rig}}(X/K)$ and the Frobenius action on it. By the claim, we can single out the set of eigenvalues that are *not* from X_U (together with their multiplicities). Since these eigenvalues are also Frobenius eigenvalues of $\mathrm{H}^1(X_S, \mathbb{Q}_\ell)$, Theorem 1.4.1 implies that they must be algebraic integers all of whose conjugates are of slopes $\geq s$. By the exactness of (3.1.6), these eigenvalues are eigenvalues of $\mathrm{H}^1_{\mathrm{rig}}(X_S/K)$. The rest of the eigenvalues of this group are from

$$\mathrm{H}^2_{\mathrm{rig},c}(X_U/K) \cong \mathrm{H}^1_{\mathrm{rig},c}(U/K, R^1 f_* \mathcal{O}^\dagger_{X_U/K}) \oplus \mathrm{H}^2_{\mathrm{rig},c}(U/K, \mathcal{O}^\dagger_{U/K}).$$

By the hypothesis, the slopes of $R^1 f_* \mathcal{O}^\dagger_{X_U/K}$ are $\geq s$, and those of $\mathcal{O}^\dagger_{U/K}$ are ≥ 0. Applying Theorem 3.1.4, we conclude that the slopes of $\mathrm{H}^2_c(X_U/K)$ are at least $\min(1, s)$. This completes the proof. □

Theorem 3.1.7. *Let $f : X \to B$ be a flat, projective morphism of irreducible, nonsingular, projective varieties over k that is generically smooth over B. Assume that*

(1) $\dim B = 1$,
(2) *there is a nonempty, Zariski open subset $U \subset B$ such that for all geometric points $b \in B$, the slopes of the rigid cohomology of $\mathrm{H}^i_{\mathrm{rig}}(X_b/K)$ are at least $s \leq 1$, for all $i > 0$, and*
(3) *Hypothesis 3.1.2 holds.*

Then for all $b \in B$ and all $i > 0$, $\mathrm{H}^i_{\mathrm{rig}}(X_b/K)$ has slope at least s.

Proof. First assume that k is an algebraic closure of a finite field. And we assume that X, f, B and all the singular fibers of f are defined over a finite field $k_0 \subset k$ of cardinality q. Then for any smooth, proper k-variety Y defined over k_0, the Newton polygon of $\mathrm{H}^i_{\mathrm{rig}}(Y/K)$ and that of the characteristic polynomial of the Frobenius ϕ^a on the étale cohomology $\mathrm{H}^i_{\text{ét}}(Y, \mathbb{Q}_\ell)$ coincide, thanks to the purity of the Frobenius eigenvalues. Therefore we can apply the analogue theorem in étale cohomology and conclude that all the Frobenius eigenvalues of all the singular fibers are divisible by q.

Let U be the smooth locus of f with complement S. Then we have an exact sequence

$$\mathrm{H}^i_c(X_U) \to \mathrm{H}^i(X) \to \mathrm{H}^i(X_S) \to \mathrm{H}^{i+1}_c(X_U) \tag{3.1.8}$$

in the étale theory. By the étale analogue of the theorem, we see $\mathrm{H}^i(X_S)$ has Frobenius eigenvalues divisible by q. If $i > 1$, then the cohomology of $\mathrm{H}^i_c(X_U)$ may be computed by the compact-supported Leray spectral sequence:

$$0 \to \mathrm{H}^1_c(U, R^{i-1} f_*\mathbb{Q}_\ell) \to \mathrm{H}^i_c(X_U) \to \mathrm{H}^2_c(U, R^{i-2} f_*\mathbb{Q}_\ell) \to 0.$$

Since $i > 1$, both the left hand side group and the right hand side group have Frobenius eigenvalues divisible by q by the analogue of the Weil conjecture. So the middle group has Frobenius eigenvalues divisible by q. Hence $\mathrm{H}^i(X)$ has Frobenius eigenvalues divisible by q.

The equation (3.1.8) also has a rigid version

$$\mathrm{H}^i_{\mathrm{rig},c}(X_U/K) \to \mathrm{H}^i_{\mathrm{rig}}(X) \to \mathrm{H}^i_{\mathrm{rig}}(X_S) \to \mathrm{H}^{i+1}_{\mathrm{rig},c}(X_U/K) \tag{3.1.9}$$

When $i = 1$, we can apply 3.1.5 above. When $i > 1$, the previous argument translates, via the zeta function and the purity, that $\mathrm{H}^i_{\mathrm{rig}}(X)$ has slope ≥ 1. By Hypothesis 3.1.2, and the hypothesis that all positive cohomology of the smooth fibers have slope ≥ 1, we conclude that $\mathrm{H}^{i+1}_{\mathrm{rig},c}(X_U/K)$ has slope ≥ 1. This forces the same result for $\mathrm{H}^i_{\mathrm{rig}}(X_S/K)$. This completes the proof. □

3.2. Generalizations in rigid cohomology

Let $g : Y \to S$ is a proper morphism of complex algebraic varieties. Then $R^i f_*\mathbb{C}$ is an algebraic constructible sheaf on S. Therefore there is a Zariski open dense subset U of S such that $R^i f_*\mathbb{C}|_U$ is locally constant. This phenomenon also has its counterpart in the ℓ-adic étale theory.

In this section, we address a similar question in rigid cohomology. Given a proper flat morphism $g : Y \to S$ of varieties over a perfect field, where S is smooth but Y may be singular, we prove that, after a finite purely inseparable base change, there is a convergent isocrystal E on an dense open subset S' of S whose fiber at a closed point $s \in S'$ computes the rigid cohomology of the fiber Y_s. The basic idea is to use "a family of simplicial resolutions". This convergent isocrystal turns out to be the key for the spread-out argument.

Note that, unlike the previous section, where it's crucial to work with overconvergent F-isocrystals, the present section works exclusively with convergent F-isocrystals, since we are not about to take cohomology on the base.

3.2.1. *The convergent topos of Ogus* [20]. In the following, we assume that k is a perfect field of characteristic p. Let $W = W(k)$ be the ring of Witt vectors of k. Let $K = W[1/p]$ be the fraction field of W.

Let X be a variety over k. An *enlargement* of X/W is an admissible (i.e., flat, locally topologically finitely presented) formal scheme $\mathfrak{Z}$ over Spf(V), plus a V-morphism $z : Z \to X$, where Z is a closed subscheme of definition of $\mathfrak{Z}$, and Z contains $(\mathfrak{Z}_k)_{\mathrm{red}}$ as a closed subscheme. (Here we are using different notations from Ogus's; in [20], our $\mathfrak{Z}_k$ is called $\mathfrak{Z}_1$, and Ogus uses $\mathfrak{Z}_0$ to denote the canonical reduction $(\mathfrak{Z}_k)_{\mathrm{red}}$.) The morphisms of enlargements are defined naturally. The category of all enlargements forms a site Enl(X/W), when equipped with the Zariski topology. The *convergent topos*, notation $(X/W)_{\mathrm{conv}}$, is then the sheaf topos of the site Enl(X/W).

If we assign to an enlargement $Z = (\mathfrak{Z}, z)$ the ring $\mathcal{O}_{X/W}(Z) = \mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})$, we get a sheaf of rings in $(X/W)_{\mathrm{conv}}$, which is denoted by $\mathcal{O}_{X/W}$. Similarly, the assignment $Z \mapsto \mathcal{O}_{X/W}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ defines a sheaf of rings, which is denoted by $\mathcal{O}_{X/W}$. (Ogus denotes it by $\mathcal{K}_{X/W}$ in [20].)

The category of coherent $\mathcal{O}_{X/W}$-crystals and coherent $\mathcal{O}_{X/K}$-crystals can be defined as in the crystalline case. A coherent $\mathcal{O}_{X/K}$-crystal in the convergent topos is also known as a *convergent isocrystal*. Those convergent isocrystals acquiring Frobenius structures ("convergent F-isocrystals") form a category which is denoted by F-Isoc(X/S).

3.2.2. Let S be an irreducible, smooth, affine scheme over k admitting a smooth formal lifting $\mathfrak{S}$ over W. Assume further that the absolute Frobenius of S also lifts to $\mathfrak{S}$.

Although the definition of the convergent site [20] does *not* allow us to evaluate a convergent sheaf on not-necessarily topologically-finite-type formal schemes over W, for crystals and isocrystals we may still do so, as pointed out by Crew [5]. Let us briefly recall how this works.

Let E be a convergent crystal on S. Then E defines a coherent locally free sheaf $\mathcal{E}$ on $\mathfrak{S}$. So it makes sense to pull back $\mathcal{E}$ to any flat formal scheme $\mathfrak{T}$ over $\mathfrak{S}$. We shall denote this pull back by $E_{\mathfrak{T}}$ as if we are evaluating the crystal on a $\mathfrak{T}$.

If E is merely a convergent isocrystal, then we cannot do the above on the nose. The upshot is that we can do this "generically": away from a codimension two formal subscheme of $\mathfrak{S}$, E can be described as a crystal tensored with $\mathbb{Q}$. Let us prove this: By our hypothesis of S, the identity morphism of S defines an enlargement $(\mathfrak{S}, S)$ of S/W. If E is a convergent isocrystal on S, then we get a locally free $\mathcal{O}_{\mathfrak{S}} \otimes \mathbb{Q}$-module $E_{\mathfrak{S}}$ on $\mathfrak{S}$. In the language of rigid analytic geometry, we get a locally free sheaf on the Raynaud generic fiber of $\mathfrak{S}$, which is also the tube $]S[_{\mathfrak{S}}$ in the sense of Berthelot. If we cover $\mathfrak{S}$ by finitely many Zariski opens $\mathfrak{U}_i$ on which $E_{\mathfrak{S}}$ is trivialized, then by removing a proper closed subset of $\mathfrak{U}_i$ we can give an integral structure of $E_{\mathfrak{S}}$. Therefore we have deduced the following lemma, which is essentially due to Crew [5], Lemma 2.3.

Lemma 3.2.3. *Let E be a convergent F-isocrystal on S. Then up on replacing S by a suitable nonempty Zariski open subset, there is a locally free $\mathcal{O}_{\mathfrak{S}}$-module with a Frobenius structure $\mathcal{E}$ on $\mathfrak{S}$, such that for any morphism of enlargements $z : (\mathfrak{Z}, Z) \to (\mathfrak{S}, S)$, we have a canonical isomorphism*

$$z^*(\mathcal{E}) \otimes \mathbb{Q} \cong E_{\mathfrak{Z}},$$

compatible with the Frobenius structures. □

In Crew [5], the object $\mathcal{E}$, together with its Frobenius, is called an "F-lattice". (Note that Crew's paper predates the construction of the convergent site.)

Let κ a perfect field that is also an S-algebra containing the function field of S. Then $(\mathrm{Spf}(W(\kappa)), \mathrm{Spec}(\kappa))$ defines a flat formal scheme over the formal scheme $\mathfrak{S}$. Note that $\mathrm{Spf}(W(\kappa))$ is generally not of finite topological type over W, so $(\mathrm{Spf}(W(\kappa)), \mathrm{Spec}(\kappa))$ is not an enlargement of S/W. But we can still pull back the locally free $\mathcal{O}_{X/W}$-module $\mathcal{E}$ back to $\mathrm{Spf}(W(\kappa))$, obtaining a locally free sheaf

$\mathcal{E}_{W(\kappa)}$. We shall refer $\mathcal{E}_{W(\kappa)} \otimes \mathbb{Q}$ as the *value* of the F-isocrystal E at the non-finite type enlargement $(\mathrm{Spf}(W(\kappa)), \mathrm{Spec}(\kappa))$. We denote it by E_κ.

3.2.4 (The Newton polygon at a perfect point)**.** Let κ be a perfect field, and let $x : \mathrm{Spec}(\kappa) \to S$ be a morphism. Then as we explained 3.2.2, we can evaluate a convergent crystal E at x. The Frobenius structure allows us to talk about the slopes and Newton polygons of E_x. Clearly, this information only depends on the image of κ in S. In this situation, one can show that the Grothendieck specialization holds:

Proposition 3.2.5 (Grothendieck, Katz)**.** *Let notations be as in 3.2.4. Let E be a convergent crystal on S. The end points of the Newton polygons of E are all the same. For any convex polygon P, the locus of the points where the Newton polygon of E is over P is Zariski closed (see e.g., [5], Theorem 1.6, Proposition 1.7).* □

Corollary 3.2.6 (Cf. Crew [5], Theorem 2.1)**.** *Let E be a convergent F-isocrystal on S. Then there is a stratification of $S = \coprod_\alpha S_\alpha$ into a disjoint union of finitely many locally closed subsets, such that the following hold true.*

(1) *The Newton polygon of E on S_α is constant.*
(2) *If $S_\alpha \subset \overline{S_\beta}$, then the Newton polygon of E on S_β is no higher than that of S_α.*
(3) *All the Newton polygons of E have the same endpoints.* □

3.2.7. Let S be as in 3.2.2. Let $f : X \to S$ be a smooth proper morphism of k-varieties. For each $i \geq 0$, then Ogus [19] has defined a unique convergent F-isocrystal $R^i f_* \mathcal{O}_{X/K}$, satisfying a series of properties. For us, the most important things to know are following.

(1) If $\mathfrak{T}, T$ is a p-adic enlargement of S/W, and

$$R^i_{f/\mathfrak{T}} = R^i f_{X/\mathfrak{T}}(\mathcal{O}_{X_T/\mathfrak{T}})$$

is the crystalline cohomology sheaf, then we have a natural isomorphism $(R^i f_* \mathcal{O}_{X/K})_{\mathfrak{T}} \cong R^i_{f/\mathfrak{T}} \otimes \mathbb{Q}$ ([19], Theorem 3.1).

(2) In particular, for any closed point $s \in S$, we have

$$\mathrm{H}_{\mathrm{cris}}(X_s/W(s)) \otimes \mathbb{Q} \cong (R^i f_* \mathcal{O}_{X/K})_s.$$

Here, $W(s)$ is the ring of Witt vectors for the residue field of s.

(3) Its formation is compatible with arbitrary finite type base change $S' \to S$.

3.2.8. For any convergent isocrystal E on S, we have defined its value at a possibly non-closed perfect point κ, and the corresponding Newton polygon at this point. When the isocrystal is $E = R^i f_* \mathcal{O}_{X/S}$ as above, what is the relation between E_κ and the crystalline cohomology of $X \otimes \kappa$? The property 3.2.7, (1) of the Ogus's isocrystal gives the answer when κ is above the generic point of S: they agree.

Lemma 3.2.9. *In the situation 3.2.7, let κ be a perfect field extension of the function field $k(S)$. Then there is a natural isomorphism between $\mathrm{H}^i_{\mathrm{cris}}(X_\kappa) \otimes \mathbb{Q}$ and $(R^i f_* \mathcal{O}_{X/S})_\kappa$.*

Proof. Since $\mathrm{Spec}(\kappa) \to S$ is flat, the local criterion of flatness implies $u : \mathrm{Spf}(W(\kappa)) \to \mathfrak{S}$ is flat as well. Using the flat base change of crystalline cohomology, Theorem 7.8 of [3], we infer that the rigid cohomology of X_κ at degree i is equal to

$$u^* R^i_{f/\mathfrak{S}}.$$

Hence

$$\mathrm{H}^i_{\mathrm{cris}}(X_\kappa) \otimes \mathbb{Q} \cong u^* R^i_{f/\mathfrak{S}} \otimes \mathbb{Q} \cong u^*(\mathcal{E}) \otimes \mathbb{Q} = (R^i f_* \mathcal{O}_{X/K})_\kappa.$$

Here pullbacks are performed with respect to the Zariski topology, and $\mathcal{E}$ is the vector bundle on $\mathfrak{S}$ we constructed in Lemma 3.2.3. Note that for our purpose we can always replace S by a Zariski open dense subset, thus we can assume at the beginning that $\mathcal{E}$ is a vector defined on all $\mathfrak{S}$. This finishes the proof. □

We include the following lemma, which is well-known (I cannot find a reference), as an illustration of the spread-out method we'll be using later.

Lemma 3.2.10. *Let X be a smooth, projective variety over k. Then the slopes of $\mathrm{H}^i_{\mathrm{cris}}(X/K)$ are all ≥ 0.*

Proof. The variety X is defined over a finitely generated field over a finite field $\mathbb{F}_q$. Therefore there is a smooth, projective morphism

$$\varphi : \mathcal{X} \to S$$

of nonsingular varieties over a finite field $\mathbb{F}_q$, and a dominant morphism $\mathrm{Spec}(k) \to S$, such that $\mathcal{X} \times_S \mathrm{Spec}(k) = X$. For any closed point s of S, the slopes of $\mathcal{X}_s$ are non-negative, since these are the slopes of the reciprocals of the zeros and poles of the zeta function of $\mathcal{X}_s$. By the Grothendieck trace formula, they are algebraic integers, whence are of slopes ≥ 0. By Proposition 3.2.5, the Newton polygon of $\mathrm{H}^i_{\mathrm{cris}}(X)$, which agrees with the Newton polygon of the Ogus isocrystal $R^i f_* \mathcal{O}_{X/K}$ at the non-finite type enlargement $(\mathrm{Spf}(W(k)), \mathrm{Spec}(k))$ by Lemma 3.2.9, is not below the Newton polygon of any closed point of S. It follows that it must be in the first quadrant, and it has slopes ≥ 0. □

3.2.11. We now turn to the main goal of this section: to define the "generic direct image" isocrystal for a singular variety. Let S be as in 3.2.2. Let N be a positive integer. Let $X_\bullet \to S$ be an N-truncated, smooth, projective, simplicial scheme over S. Let $E_{m,n}$ be the convergent F-isocrystals $R^n f_*(\mathcal{O}_{X_m/K})$. According to the general nonsense, there is a spectral sequence with

$$E_1^{m,n} = E_{m,n} \Rightarrow R^{m+n} f_{\bullet,*}(\mathcal{O}_{X_\bullet/K}).$$

Lemma 3.2.12. *In the situation above, $R^m f_{\bullet,*}(\mathcal{O}_{X_\bullet/K})$ is a convergent F-isocrystal for all $m < N/2$.*

Proof. Since coherent isocrystals form a fully faithful abelian subcategory of $\mathcal{O}_{S/K}$-modules, we infer that $E_s^{i,j}$ will be a convergent isocrystal if the three $E_{s-1}^{i,j}$, $E_{s-1}^{i+s-1,j-s+2}$, and $E_{s-1}^{i-s+1,j+s-2}$ are convergent isocrystals. We already know that, for all $i < N$, $E_1^{i,j}$ are convergent isocrystals by the construction of the spectral sequence. Therefore, some simple counting shows that whenever $2m < N$, all $i \geq 0$, all $s < m+2$, $E_s^{i,m-i}$ are in the "good" range. On the other hand, the spectral sequence we are discussing is a first quadrant spectral sequence. So $E_{m+1}^{i,m-i} = E_\infty^{i,m-i}$. □

3.2.13. *Hypotheses.* Let S be as in Situation 3.2.2. Let $g : Y \to S$ be a k-morphism where Y is a possibly singular projective variety over k. Assume that there is a smooth, projective, N-truncated simplicial S-variety $f : X_\bullet \to S$, and an S-morphism $h : X_\bullet \to Y$, that is an N-truncated projective hypercovering. In this situation, in the view of Lemma 3.2.12, for $m < N/2$, we denote $\mathcal{R}^m$ to be the convergent F-isocrystal $R^m f_{\bullet,*}(\mathcal{O}_{X_\bullet/K})$.

The above hypothesis is fairly strong, and is surely not to be satisfied for an arbitrary $g : Y \to S$. But in the applications we have in mind we can always make some arrangements to force it to happen.

Lemma 3.2.14. *Under the hypotheses assumed in 3.2.13, for any perfect field κ and any κ-point $x : \mathrm{Spec}(\kappa) \to S$, we have*

$$\mathrm{H}^m_{\mathrm{rig}}(Y/W(\kappa)) \cong (\mathcal{R}^m)_\kappa.$$

In particular, the mth rigid Betti number for Y is constant on S. Moreover, there is a stratification of S into locally closed subsets on which the Newton polygon of $\mathrm{H}^m_{\mathrm{rig}}(Y)$ is constant; and if $S_\alpha \subset \overline{S_\beta}$, then the Newton polygon in S_α is no lower than that on S_β.

Proof. By the hypotheses, for any x as in the statement, $X_{\bullet,x} \to Y_x$ is a proper hypercovering. By Tsuzuki's theorem on cohomological descent [27], and Berthelot's comparison theorem between rigid and crystalline cohomology groups for smooth proper varieties ([20], Theorem 0.7.7), the value $(\mathcal{R}^m)_x$ equals the rigid cohomology $\mathrm{H}^m_{\mathrm{rig}}(Y_x)$ for all "perfect" points $x : \mathrm{Spec}(\kappa) \to S$. Since $\mathcal{R}^m$ is a convergent isocrystal on S, the assertions now follows from Corollary 3.2.6. □

Corollary 3.2.15. *Let $k \subset k'$ be an extension of perfect field. Let K, resp. K' be the fraction field of $W(k)$, resp. $W(k')$. Let Y be a proper, finite type, separated scheme over k. Then for each i, one has a natural isomorphism*

$$\mathrm{H}^i_{\mathrm{rig}}(Y/K) \otimes_K K' \cong \mathrm{H}^i_{\mathrm{rig}}(Y'/K')$$

compatible with Frobenius structures.

Proof. In Lemma 3.2.14, take $\mathfrak{S} = \mathrm{Spf}(W)$, $S = \mathrm{Spec}(k)$. □

Proposition 3.2.16. *Let S be as in 3.2.2. Let N be a fixed positive integer. Let $f : Y \to S$ be a flat, projective morphism. Then there exist a Zariski open subset S' of S, a purely inseparable, finite morphism $T \to S'$, an N-truncated simplicial smooth, projective T-variety $X_\bullet$, and a morphism $X_\bullet \to Y_T$ that is a projective hypercovering.*

Proof. Let $L = k(S)$ and L^{perf} be the perfection of L. Then $Y_{L^{\mathrm{perf}}}$ is a typically singular, projective variety on L^{perf}. We can apply de Jong's alteration [6] to construct a projective, N-truncated hypercovering

$$h_{L^{\mathrm{perf}}} : X_{\bullet,L^{\mathrm{perf}}} \to Y_{L^{\mathrm{perf}}}.$$

by smooth, projective L^{perf}-varieties. The varieties X_m, $m = 1, 2, \ldots, N$, are necessarily defined over a finite, purely inseparable extension of L. So upon shrinking S

and replacing S by a purely inseparable finite covering, and a further shrinking, we can assume that the projective hypercovering has a model $X_\bullet \to Y_T$ on T, where T is a variety that is a finite purely inseparable covering of some nonempty Zariski open subset S' of S. $\square$

Definition 3.2.17. In the situation of Proposition 3.2.16, assume $N > 4 \dim Y$. Then for any $0 \leq m \leq 2 \dim Y$, the F-isocrystals $\mathscr{R}^m$ constructed in 3.2.13 are called "generic direct images" for $Y_T \to T$.

3.3. Semistable degeneration of slopes

In this section we prove an analogue of Theorem 1.4.1 for log crystalline cohomology groups, assuming the degeneration has semistable singularities. We make the following hypotheses.

3.3.1. Let k be a finite field. Let W be the ring of Witt vectors of k. Let K be the fraction field of W. Let $f : X \to B$ be a smooth, projective morphism of nonsingular k-varieties. Assume that f is smooth away from a closed point $b \in B$, and $f^{-1}(b)$ is a simple normal crossing divisor in X. Let $0 \leq s \leq 1$ be a rational number. Assume that there is a Zariski open subset $U \subset B$ such that for all $t \in B$, the slopes of the crystalline cohomology $\mathrm{H}^i_{\mathrm{cris}}(X_t) \otimes_W K$ are all $\geq s$.

The main result of this section is the following.

Proposition 3.3.2. *In Situation 3.3.1, the slopes of* $\mathrm{H}^i_{\mathrm{rig}}(X_b/K)$ *are at least* s.

The proof of Proposition 3.3.2 resembles the strategy used in the étale case. The first is to relate the geometric condition on the crystalline cohomology of general fibers to the "nearby fiber" of the singular fiber. This is formally achieved by the log crystalline cohomology of a log scheme, and one needs to use the log-convergent theory of Shiho [26], [24]. The second is to relate the log crystalline cohomology of the nearby fiber with the singular fiber; to this end we need a sort of the local invariant cycle theorem. This is provided in the present case by the p-adic Clemens-Schmid exact sequence of Chiarellotto-Tsuzuki [4].

We first recall an analogue of the Grothendieck specialization theorem in Shiho's context of log convergent site. In our special situation, this can be reduced to the results proven by Crew. For convenience, let us present a proof.

Lemma 3.3.3. *Let notations be as in Situation 3.3.1. We regard B as a fine, saturated, log scheme of dimension 1, equipped with the divisiorial log structure from a closed point b, whose log structure is denoted by M. Let (E, Φ) be a log convergent F-isocrystal on (B, M). Let P be the generic Newton polygon of the isocrystal (E, Φ). Then the Newton polygon of (E, Φ) at b is no lower than P.*

Proof. Without loss of generality we assume b is a k-valued point. Otherwise we can always make a finite field extension. Since we may freely shrink, we can fix a smooth formal lifting $\mathfrak{B}$ of B (whose Raynaud generic fiber is denoted by $\mathfrak{B}^{\mathrm{an}}$), together with a lifting σ of the absolute Frobenius, and a lifting $\mathfrak{b}$ of b. Then any log convergent isocrystal on (B, M) is the same as a log-∇-module on $\mathfrak{B}$: a coherent $\mathcal{O}_{\mathfrak{B}}[1/p]$-module $\mathcal{E}$ together with a K-linear integrable log-connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_K \Omega^1_{\mathfrak{B}^{\mathrm{an}}/K}(\log \mathfrak{b}^{\mathrm{an}}).$$

When the isocrystal has a Frobenius structure, the coherent sheaf also has a Frobenius structure, i.e., an isomorphism $\Phi : \sigma^*\mathcal{E} \to \mathcal{E}$. Since $\mathfrak{B}$ is of dimension one, the process of "clearing the denominator" can always be done, and we can assume that $\mathcal{E}$ is actually coherent over $\mathcal{O}_{\mathfrak{B}}$. By replacing Φ with $p^m\Phi$, we can even assume that the isomorphism Φ is defined integrally.

Now let E be as in the statement, and let $\mathcal{E}$ be the $\mathcal{O}_{\mathfrak{B}}$-module with a Frobenius structure. In the language of Crew, $\mathcal{E}$ is an F-lattice on $\mathfrak{B}$. Applying the Grothendieck specialization theorem in the form of [5], Theorem 2.1, we find that the Newton polygon at the special point b is no higher than the generic one. This completes the proof. □

Let us go back to Situation 3.3.1.

3.3.4 (The "Nearby fiber")**.** There is a natural log structure on the special fiber X_b, namely, the restriction of M to X_b. We shall denote this log scheme by (X_b, M_b). This log scheme is the analogue of the "nearby fiber" in the present situation. It's log-crystalline cohomology will be thought as the analogue of the "limiting Hodge structures" in the Hodge setting.

3.3.5 (The log convergent direct images)**.** The theory of log convergent topos developed by Shiho gives us a log convergent F-isocrystal $R^i f_{\mathrm{logconv}*}\mathcal{K}_{X/K}$ computing the log crystalline cohomologies. Here, following the notation of Shiho, we use $\mathcal{K}$ to denote the log convergent structure sheaf inverting p. As pointed out by [18], Remark 2.8 and [25], §1, these are indeed coherent $\mathcal{K}_{(B,M)/W}$-modules compatible with base change. Shiho's theory also identifies the log-convergent cohomology of log-convergent F-isocrystals with their corresponding rigid cohomology. See [24], Theorem 2.4.4.

Lemma 3.3.6. *The slopes of* $\mathrm{H}^i((X_b, M_b), \mathcal{K})$ *are at least* s. □

Proof. Applying Lemma 3.3.3, we conclude that the slopes of the Newton polygon of $R^i f_{\mathrm{logconv}*}\mathcal{K}_{X/K}$ at the point b is at least s. □

To conclude the proof of Proposition 3.3.2, the final input is the following p-adic analogue of the Clemens-Schmid exact sequence.

Theorem 3.3.7 (Chiarellotto-Tsuzuki [4])**.** *There is an exact sequence*

$$\mathrm{H}^i_{X_b,\mathrm{rig}}(X) \to \mathrm{H}^i(X_b) \to \mathrm{H}^i((X_b, M_b), \mathcal{K}).$$

By the "purity theorem" in rigid cohomology over a finite field (see e.g., [11], Lemma 2.1) the slopes of $\mathrm{H}^i_{X_b,\mathrm{rig}}(X)$ at at least 1. By Lemma 3.3.6, the slopes of $\mathrm{H}^i((X_b, M_b), \mathcal{K})$ are at least s. Since $s \leq 1$, the exactness sequence in Theorem 3.3.7 implies all the slopes of X_b are at least s. This finishes the proof of Proposition 3.3.2. □

3.4. Spreading-out

This section uses the results in §3.2 to prove Proposition 1.3.6 and Proposition 1.3.7.

Theorem 3.4.1. *Suppose that Proposition 1.3.6 holds for varieties over finite fields. Let* $f : X \to B$ *be a projective morphism of nonsingular varieties over a perfect field* k*. Suppose that the slopes of a general fiber of* f *are* $\geq s$*. Then the same holds for all fibers.*

Proof. By Corollary 3.2.15, the problem is of geometric nature, so we can assume k is an algebraically closed field. In particular, the semistable components of the singular fibers are defined over k.

The field k can be realized as an algebraically closed field containing the function field of a smooth variety S defined over some finite field $\mathbb{F}_q$. By shrinking S and perform finite base changes, the varieties B and X admit smooth, projective models $\mathcal{B} \to S$ and $\mathcal{X} \to S$ respectively. By a further restriction, we can assume f is defined over S. We use the same letter f to denote this S-morphism $\mathcal{X} \to \mathcal{B}$. Each point $x \in B$ defines a section $\sigma_x : S \to \mathcal{B}$. If f has semistable fibers, then by shrinking further (note that f has only finitely many singular fibers over k), we can assume that for any closed point t of S, the variety $\mathcal{X}_{\sigma_x(t)}$ is geometrically semistable.

For each $t \in S$, $f_t : \mathcal{X}_t \to \mathcal{B}_t$ is defined over a finite field. So we can apply the finite field version of the theorem, proven in §3.1, to conclude that for all closed points $b \in \mathcal{B}_t$, the slopes of the rigid cohomology of $\mathcal{X}_b$ are $\geq s$ in degree ≥ 1. Letting s range all closed points of S, we conclude that for all closed points $b \in \mathcal{B}$, the rigid cohomology groups $\mathrm{H}^i(\mathcal{X}_b)$ have slopes $\geq s$ for all $i \geq 1$.

Now consider the family variety $f_{\sigma,x} : \mathcal{X}_{\sigma_x(S)} \to \sigma_x(S)$. This is a flat family of singular varieties whose geometric generic fiber is X_x, and we know that all the rigid cohomology in positive degree of all fibers of this morphism are of slopes at least s. By restricting to an open subset S'_x of $\sigma_x(S)$ and passing to a finite, purely inseparable extension of T_x, we have constructed in §3.2 the "generic direct image" $\mathcal{R}^j$ for the morphism $f_{\sigma,x}$. This convergent F-isocrystal computes the degree j rigid cohomology groups of all perfect points. It is of slopes $\geq s$ on all closed points, hence by Corollary 3.2.6, its Newton polygon at the perfect point x are of slopes $\geq s$. By Lemma 3.2.14, we know this Newton polygon is precisely the Newton polygon of $\mathrm{H}^i_{\mathrm{rig}}(X_x/W(k)[1/p])$. This completes the proof. □

The same argument shows that, in the semistable case, Proposition 1.3.7 also follows from its finite field incarnation, Proposition 3.3.2. (We have indicated in the proof above how to make modifications for the semistable case.)

CHAPTER 4

A few geometric applications

We end this article by giving some applications of the results proven above. The first is about the non-degeneration phenomenon for supersingular abelian varieties. This result, due to Oort, is of course well-known before this article. The second is about the degenerations of surfaces. The results there are known before for K3 surfaces, but I don't know where it is stated explicitly for other "supersingular surfaces". The last application is more serious: we prove that the smallest slope of a general hypersurface in a product of projective spaces $\prod_{i=1}^{r}\mathbb{P}^{n_i}$ is zero, if the multi-degree $(d_1,\ldots,d_r)$ satisfies $d_i \geq n_i + 1$. When $r = 1$ this is due to Katz (later greatly generalized by Illusie). For the ease of typing I expose the proof only for $r = 2$, but for general r the method obviously generalizes. The method is quite general, so it is expected to be applicable in other situations.

4.1. Degeneration of supersingular abelian varieties

Oort proves that supersingular abelian varieties are contained in the interior of any moduli of polarized abelian varieties. I wish to redraw this conclusion by applying the semicontinuity theorem of the charge function. Standard reduction allows us to treat the case of abelian varieties defined over finite fields.

Let Δ be the spectrum of the strict henselian localization of an algebraic curve C over $\mathbb{F}_q$. Let Δ^* be the generic point of Δ. Let $\varphi : \mathcal{M}^* \to \Delta^*$ be a family of abelian varieties of dimension g, obtained from restricting a family of abelian varieties over the algebraic curve C. Assume that the family has *unipotent monodromy* $\gamma = \exp(2\pi i N)$, N being a nilpotent operator. In fact, it's follows from the monodromy-weight conjecture (proven by Deligne for varieties over equal characteristics in [7]) that $N^2 = 0$.

Then: Néron's theory asserts that there is a family of group varieties $\mathcal{M}$ over Δ that restricts to the original family $\mathcal{M}^*$, and whose special fiber M fits into an extension of group varieties:

$$1 \to T \to M \to B \to 1 \qquad (\dagger)$$

where T is a product of $\mathbb{G}_{\mathrm{m}}$, and B is an abelian variety.

It's well known that N puts a *monodromy weight filtration* $W_\bullet(H)$ on the first cohomology $H = \mathrm{H}^1(X, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^{2g}$ of nearby fibers. Moreover $W_{-1}(H) = 0$, $W_2(H) = H$. Then we know:

(1) The dimension of T equals the dimension of $\mathrm{gr}_0(H)$ with respect to the monodromy weight filtration.
(2) The dimension of B equals the dimension of $\mathrm{gr}_1(H)$ with respect to the monodromy weight filtration.

Knowing the dimension of $\mathrm{gr}_0(H) \cong \mathrm{gr}_1(H) \otimes \mathbb{Q}_\ell(1)$ amounts to determining the torus factor T, as tori have no moduli. Knowing the the polarized integral Hodge structure $\mathrm{gr}_1(H)$ of weight 1 amounts to recovering the abelian variety factor B. Therefore the information of the representation $\mathrm{H}^1(\mathcal{M}^*, \mathbb{Q}_\ell)$ tells how the two structures intertwine with each other, and ultimately recovers the extension (†).

However, if the family $\mathcal{M}$ is from a family of supersingular abelian varieties, i.e., the $r(\mathrm{H}^1(\mathcal{M}_t, \mathbb{Q}_\ell)) \leq |q|^{1/2}$, for general t in the family, then by the semicontinuity of charges, we infer that $r(\mathrm{H}^1(\mathcal{M}^*, \mathbb{Q}_\ell)) \leq |q|^{1/2}$, i.e., the monodromy weight filtration has no weight zero part! Therefore the family cannot degenerate.

4.2. Degeneration of supersingular surfaces

A smooth, projective surface S with $b_1(S) = 0$ over $\overline{\mathbb{F}}_q$ is called *supersingular* (in the sense of Michael Artin) if $\mathrm{H}^2(S)$ has only one slope (which is necessarily 1). S is called *supersingular in the sense of Shioda* if $b_2(S) = \varrho(S)$, where $\varrho(S)$ is the Néron-Severi rank of S. The two are equivalent assuming the Tate conjecture.

Unirational surfaces are one huge class of supersingular surfaces. There are also non-unirational supersingular surfaces, such as Godeaux surfaces associated to some special quintics, but no known examples are simply connected. (Shioda suspects that any simply connected supersingular surface is unirational. If one wishes to disprove this conjecture, one may try to prove some reductions of a general Barlow surface are not unirational.)

Theorem 1.4.1 allows us to say something on the degenerations of supersingular surfaces. We merely give a few illustrations in this regard. Let $\mathcal{S} \to \Delta$ be a semistable degeneration of surfaces. Assume that the general member of the family is a supersingular surface.

Lemma 4.2.1. *The homology of the dual complex of the central fiber S_0 must have vanishing* H_2.

Proof. One can compute the cohomology of the central fiber S_0 by means of the Mayer-Vietoris spectral sequence. The usual theory of weights implies this spectral sequence degenerates at the E_2 stage. If F_i are the components of S_0, then E_1 of the spectral sequence reads

$$\begin{array}{ccccccc}
\oplus\, \mathrm{H}^4(F_i) & & 0 & & & & \\
0 & & 0 & & & & \\
\oplus\, \mathrm{H}^2(F_i) & \to & \oplus\, \mathrm{H}^2(F_i \cap F_j) & & 0 & & \\
0 & & 0 & & 0 & & \\
\oplus\, \mathrm{H}^0(F_i) & \to & \oplus\, \mathrm{H}^0(F_i \cap F_j) & \to & \oplus\, \mathrm{H}^0(F_i \cap F_j \cap F_k) \to & & 0
\end{array}$$

The horizontal arrows are d_1. The bottom row computes the homology of the dual complex. Were the H_2 of the dual complex nonzero, there would be a weight zero factor in the final $\mathrm{H}^2(S_0)$. However weight zero Weil numbers are never divisible by q, whence cannot have slope 1. □

The claim recovers the classical fact that any supersingular K3 surface does not have a “Type II” semistable degeneration over fields of positive characteristics.

4.3. Hypersurfaces in a product of projective spaces

Recall that the if V is a vector space over a field k of dimension n, and $\{e_1, \dots, e_n\}$ is a basis of V, then the *Koszul complex* is

$$0 \to k \to \bigwedge^1 V \to \bigwedge^2 V \to \cdots \to \bigwedge^n V \to 0.$$

Here $d(\xi) = \sum_j (-1)^j \xi \wedge e_j$. This is an acyclic complex.

If we modify the Koszul complex by omitting the $\bigwedge^0$ and $\bigwedge^n$ terms, then we get the simplicial cochain complex of the dual complex of the generic $(n+1)$-hyperplane arrangement. The exactness of the Koszul complex tells us that in fact the nonzero cohomology of the dual complex lies in the top and bottom degrees. In the following we generalize this phenomenon to hypersurfaces in a product of projective spaces.

Below, we let V and W be two vector spaces over $\mathbb{C}$ of dimension m and n respectively. Let $d \geq m$ and $e \geq n$ be integers. Let $P = \mathbb{P}(V) \times \mathbb{P}(W)$. We shall cook up a "modified" Koszul complex that computes the cohomology of the dual complex of a "multi-hyperplane arrangement" of bidegree (m, n).

4.3.1. We define H to be the hypersurface of P cut out by the equation

$$u_1 \cdots u_m \cdot v_1 \cdots v_n$$

where u_i and v_j are homogeneous coordinates on $\mathbb{P}(V)$ and $\mathbb{P}(W)$ respectively. Let Γ be the dual complex of H. If $\gamma = (i_1, \ldots, i_r; j_1, \ldots, j_s)$ is an r-cell of Γ, we use H_γ to denote the corresponding subvariety of H defining γ.

Let $F(u)$ (resp. $G(v)$) be general homogeneous equations in u (resp. v) such that $\{F(u) = 0\} \times \mathbb{P}(W)$ (resp. $\mathbb{P}(V) \times \{G(v) = 0\}$) meets H_γ transversely for all γ. Such a hypersurfaces may not be defined over the base field; but by Bertini's theorem, it does exist after a finite extension of k. Let X' be the hypersurface in P defined by the equation

$$F(u)u_2 \cdots u_m \cdot G(v)v_2 \cdots v_n = 0.$$

Lemma 4.3.2. *The simplicial cochain complex $S^*(X')$ of the dual complex of X' is the same as that of Γ up to degree $\dim X' - 1$, and the rank of $S^{\dim X'}(X)$ is bigger than that of $S^{\dim X'}(\Gamma)$.*

This is because the number of top cell can only become larger when the degree increases, since there are more than one intersection points with a curve.

Theorem 4.3.3. *Let X' be as above. Then the dual complex of X' has nonvanishing cohomology at degree $m + n - 3$.*

Proof. Since the simplicial complex of the dual complex of X' is the same as a multi-hyperplane arrangement up to top degree, and has more elements in the top degree, it suffices to consider the case when X' is a multi-hyperplane arrangement. In this case the simplicial complex of the dual complex can be described as the *modified Koszul complex K'* as follows: the ith term of the complex K' is

$$K'_i = \bigoplus_{a+b=i, a<m, b<n} \bigwedge^a V \otimes \bigwedge^b W$$

and the mapping $d : K'_i \to K'_{i+1}$ is the composition

$$K'_i \subset K_i \xrightarrow{d} K_{i+1} \to K'_{i+1}.$$

It's not hard to check that $d^2 = 0$, and that (K', d) computes the cohomology of the dual complex of the multi-hyperplane arrange in $\mathbb{P}(V) \times \mathbb{P}(W)$.

By the Koszul description of the dual complex, it suffices to prove the mapping

$$\bigwedge^{m-1} V \otimes \bigwedge^{n-2} W \oplus \bigwedge^{m-2} V \otimes \bigwedge^{n-1} W \to \bigwedge^{m-1} V \otimes \bigwedge^{n-1} W$$

is not surjective. By taking duals, this reduces to proving the mapping

$$V \otimes W \to \left(\bigwedge^2 V \otimes W\right) \oplus \left(V \otimes \bigwedge^2 W\right)$$

is not injective. This is indeed the case: consider the element

$$\left(\sum (-1)^i e_i\right) \otimes \left(\sum (-1)^j f_j\right).$$

It's image is

$$\begin{aligned} &\left(\textstyle\sum_{i,s} (-1)^i (-1)^s e_i \wedge e_s\right) \otimes \left(\sum (-1)^j f_j\right) \\ \pm &\left(\textstyle\sum (-1)^i e_i\right) \otimes \left(\sum_{j,\ell} (-1)^j f_j \wedge f_\ell\right) = 0. \end{aligned}$$

This proves the theorem. □

We can now state the main result of this subsection.

Theorem 4.3.4. *Let X be a general hypersurface in $\mathbb{P}(V) \times \mathbb{P}(W)$ of degree (d, e) with $d \geq \dim V$, $e \geq \dim W$. Assume $\dim X \geq 2$. Then the rigid cohomology of X has a slope equal to zero.*

Proof. Since the parameter space is defined over a finite field, and is rational, and since the slope ≥ 1 condition specializes, it suffices to prove this over a finite field. Let s be the smallest slope of the middle cohomology of X. So we can use the method of étale cohomology. Choose a 1-parameter degeneration of X to a hypersurface X' of the type above, then by Theorem 1.4.1, we infer that the smallest slope of X' is at least s. We claim that the middle cohomology of X' has a piece of weight zero, and then the corollary would follow. This is because we can compute the cohomology of X' by means of the Mayer-Vietoris spectral sequence, and the complex $(E_1^{j,0}, d_1)$ is precisely computing the cohomology of the dual complex of X'. By Theorem 4.3.3, we conclude that $E_2^{\dim X',0} \neq 0$. Since the spectral sequence is E_2-degenerate, we have

$E_2^{\dim X',0} = E_\infty^{\dim X',0}$. It follows that $\mathrm{H}^{\dim X'}(X', \mathbb{Q}_\ell)$ has a Frobenius eigenvalue of weight zero. An algebraic number of weight zero, i.e., all of whose complex conjugates are of length 1, is necessarily a root of 1. So its p-adic slope is always 0 for all embeddings $\iota : \overline{\mathbb{Z}} \to \overline{\mathbb{Q}}_p$. This completes the proof. □

Remark 4.3.5. A similar result for complete intersections in a single projective space $\mathbb{P}^n$ has been proven by Katz [SGA7$_{\mathrm{II}}$, Exposé XX] (later generalized by Illusie [13]: Illusie managed to prove that a general complete intersection is *ordinary).*

Theorem 4.3.4, when combined with Lemma 4.3.7 below, implies the following result:

Corollary 4.3.6. *A general multi-hypersurface X of bidegree (d, e) with $d \geq \dim V$, $e \geq \dim W$, is* not *uniruled.*

For non-Fano complete intersections in a projective space, a similar result has been proven by Riedl and Woolf [22]. Their method seems to be geometric, whereas ours is purely cohomological. Note also that the varieties in Corollary 4.3.6 are of Picard number 2.

Lemma 4.3.7. *Let X be a geometrically connected, nonsingular variety over a finite field k of q elements. Assume that the dimension of X equals n, and that X is geometrically uniruled. Then the Frobenius eigenvalues of $\mathrm{H}^n_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ are all algebraic integers divisible by q.*

The proof of the lemma is given in the end of this section.

Proof of 4.3.6 (assuming 4.3.7). We use the fact that the parameter space of multi-hypersurfaces is uniruled, so the problem immediately reduces to a finite field. If X is as in the hypothesis of Corollary 4.3.6 over a finite field k, then by 4.3.4, its Newton polygon has at least one piece of slope 0 hence cannot be uniruled by the lemma. □

Remark 4.3.8. Recall that saying X is geometrically uniruled amounts to declaring the existence of a proper, dominant, *rational* map

$$\varphi : \mathbb{P}^1 \times M \to X$$

with $\dim M = n - 1$ (after possibly performing an extension of k, which will not alter the result). By performing an alteration, we may assume that M is smooth and projective over k.

Remark 4.3.9. Assuming the resolution of singularity in characteristic p, the lemma is quickly proven: for any $(n-1)$-dimensional, smooth, projective variety M, the Künneth formula of cohomology implies that the Frobenius eigenvalues of $\mathrm{H}^n(\mathbb{P}^1 \times M)$ are all divisible by q. The same holds for any smooth, projective varieties that is birational to $\mathbb{P}^1 \times M$, since birational modifications do not change the "coniveau ≥ 1" condition. Let $P \to \mathbb{P}^1 \times M$ be a projective, birational morphism, then the composition $P \to \mathbb{P}^1 \times M \to X$ embeds the cohomology of X into that of P (Poincaré duality), which shows the Frobenius eigenvalues of X at middle degree are divisible.

Without the resolution of singularity, the proof becomes a bit lengthy, but is still fairly standard. (Note that alterations generally change coniveau, so are not handy in our situation.) We shall present the proof of it in the rest of this secition.

4.3.10. *Notations.* Let Y be the closure of the graph of φ. Below, to save ink, we write $\mathrm{H}^i_T(S)$ for $\mathrm{H}^i_{T_{\overline{k}}}(S_{\overline{k}}, \mathbb{Q}_\ell)$ and $\mathrm{H}^i(S)$ for $\mathrm{H}^i_\emptyset(S)$. We also use pr_i to denote the projections to the ith factor on $X \times \mathbb{P}^1 \times M$, and use $\cdot$ to denote the cup product on cohomology.

Lemma 4.3.11. *For any $c \in \mathrm{H}^n(X)$, the element $\mathrm{pr}_1^*(c) \cdot [Y]$ is nonzero.*

Proof. In fact, let $c^\vee$ be the Poincaré dual to c. Then $c \cdot c^\vee = [x]$ where x is a point class on X. Were the class zero, then

$$0 = \mathrm{pr}_1^*(c) \cdot [Y] \cdot \mathrm{pr}_1^*(c^\vee) = [Y] \cdot [\mathrm{pr}_1^{-1}(x)].$$

since φ is proper and dominant, for a general $x \in X$, $\varphi^{-1}(x)$ is a well-defined subvariety of $\mathbb{P}^1 \times M$, hence of $Y \subset X \times \mathbb{P}^1 \times M$. Its class will represent the right hand side of the displayed equation. However, the cohomology class of a subvariety is never zero (e.g., its intersection with an ample class is nonzero). This is a contradiction. □

By the construction of the class $[Y]$ and cup product, the cohomology class $c \cdot [Y]$ naturally lands in $\mathrm{H}^{3n}_Y(X \times \mathbb{P}^1 \times M)(n)$, the cohomology group with supports in Y. So Claim 4.3.11 gives an injective map

$$\Phi : \mathrm{H}^n(X) \hookrightarrow \mathrm{H}^{3n}_Y(X \times \mathbb{P}^1 \times M)(n).$$

By purity ("the Thom isomorphism"), we have

$$\mathrm{H}^{3n}_Y(X \times \mathbb{P}^1 \times M) \cong \mathrm{H}^n(Y)^*(-2n);$$

and, under the Poincaré duality, we have

$$\mathrm{H}^n(\mathbb{P}^1 \times M) \cong \mathrm{H}^n(\mathbb{P}^1 \times M)^*(-n), \quad \mathrm{H}^n(X)^* \cong \mathrm{H}^n(X)(n).$$

Therefore, to prove the lemma, it suffices to prove that all the generalized Frobenius eigenvectors of $\mathrm{H}^n(Y)$ whose eigenvalues are not divisible by q are killed by the transpose of the map $\Phi(-n)$.

Proof of 4.3.7. First we set up some notations. Let $p = \mathrm{pr}_{2,3}|_Y$ be the projection to $\mathbb{P}^1 \times M$. Then p is a birational morphism. Let U be the maximal open subset of $\mathbb{P}^1 \times M$ such that p restricts to an isomorphism on $p^{-1}(U)$. Let E be the complement of U in Y; and let Z be the complement of U in $\mathbb{P}^1 \times M$. Finally, let V be the complement $(X \times \mathbb{P}^1 \times M) \setminus Y$.

There is an exact sequences

$$\mathrm{H}^n_Z(\mathbb{P}^1 \times M) \to \mathrm{H}^n(\mathbb{P}^1 \times M) \to \mathrm{H}^n(U) \to \mathrm{H}^{n+1}_Z(\mathbb{P}^1 \times M)$$

By the Künneth formula, the Frobenius eigenvalues of $\mathbb{P}^1 \times M$ at degree n are divisible by q. By Lemma 2.0.11, the Frobenius eigenvalues of $\mathrm{H}^{n+1}_Z(\mathbb{P}^1 \times M)$ are divisible by q as well. Hence the Frobenius eigenvalues of $\mathrm{H}^n(U)$ are divisible by q.

We also have another exact sequence

$$\mathrm{H}^n_E(Y) \to \mathrm{H}^n(Y) \to \mathrm{H}^n(U).$$

So the only Frobenius eigenvalues of the middle that may not be divisible must come from $\mathrm{H}^n_E(Y)$. Here we cannot use Lemma 2.0.11 any more, since Y is generally singular. To take care of the eigenvalues of $\mathrm{H}^n_E(Y)$, we use the following exact sequence

$$\mathrm{H}^{n-1}(V) \to \mathrm{H}^n_E(Y) \to \mathrm{H}^n_E(X \times \mathbb{P}^1 \times M).$$

By Lemma 2.0.11 again, the right hand side of the displayed sequence has Frobenius eigenvalues divisible by q. So we turn to study $\mathrm{H}^{n-1}(V)$. It is handled by the exact sequence

$$\mathrm{H}^{n-1}(X \times \mathbb{P}^1 \times M) \to \mathrm{H}^{n-1}(V) \to \mathrm{H}^n_Y(X \times \mathbb{P}^1 \times M).$$

By Lemma 2.0.11, the only "bad" eigenvalues are from $\mathrm{H}^{n-1}(X \times \mathbb{P}^1 \times M)$. But the composition

$$\mathrm{H}^{n-1}(X \times \mathbb{P}^1 \times M) \to \mathrm{H}^{n-1}(V) \to \mathrm{H}^n_E(Y) \to \mathrm{H}^n(Y) \to \mathrm{H}^n(X)$$

must be the zero map, since by the Weil conjecture the source is pure of weight $n - 1$, whereas the target is pure of weight n. $\square$

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