In this note I want to expose a proof of the following theorem.

THEOREM. *Let D be a period domain of a complex Hodge structure. Then the holomorphic sectional curvature (with respect to the invariant metric) of any horizontal tangent vector is bounded from above by a positive constant.*

The proof presented here is, in my opinion, much easier than the proofs that appeared in the literature. It is based on the calculation of the curvatures of Hodge bundles.

1. The curvature of Hodge bundles. Let M be a complex manifold that admits a complex variation of Hodge structure $(\mathbf{H}, F^\bullet(\mathbf{H}), S, \nabla)$ of weight 0. Then the associated *Hodge bundle* is the graded Higgs bundle $(\mathscr{E} = \bigoplus_p \mathscr{E}^p, \theta)$, where $\mathscr{E}^p = F^p\mathbf{H}/F^{p+1}\mathbf{H}$, and the Higgs field $\theta : \mathscr{E}^p \to \Omega^1_M \otimes \mathscr{E}^{p-1}$ is induced by the second fundamental form of the connection ∇ (thanks to the infinitesimal period relation). The sesquilinear form S induces a positive definite Hermitian metric $\bigoplus_p (-1)^p S$ on $\mathscr{E}$ (the second Hodge-Riemann bilinear relation), and is called the *Hodge metric* on $\mathscr{E}$.

Using the method from Hermitian differential geometry, and the flatness of ∇, one can calculate the curvature of the Hodge bundle. On the direct summand $\mathscr{E}$, for any $(1,0)$ vector fields u, v on M and any sections e, e' of $\mathscr{E}^p$, we have

$$(\dagger) \qquad \langle \Theta(u,\bar{v})e, e'\rangle_p = \langle \theta^p_u e, \theta^p_v e'\rangle_{p-1} - \langle (\theta^{p+1}_u)^* e, (\theta^{p+1}_v)^* e'\rangle_{p+1}.$$

Here Θ is the curvature operator, θ^* is the metric adjoint of the Higgs field, and $\langle \bullet, \bullet \rangle_p$ is the Hodge metric restricting to $\mathscr{E}^p$.

We shall not present the proof of this, as it is exposed in perfectly in Schmid (1973) "Variation of Hodge structure: the singularities of the period mapping". Invent. Math., 22, 211–319.

2. The curvature of the horizontal tangent bundle. The above result allows us to calculate the holomorphic sectional curvature of the horizontal subbundle $T^h(D)$ of the period domain. Let

$$(H, F^p(H), S)$$

be a complex Hodge structure. Then we use:

- G_0 to denote the real Lie group consisting complex linear transformations on H that preserves the sesquilinear form S;
- V to denote the subgroup of G_0 consisting elements preserving the Hodge flag. Then $D = G_0/V$ is the period domain attached the Hodge structure H.

Since V preserves the Hodge filtration and the sesquilinear form, it preserves the conjugate Hodge filtration, hence the Hodge decomposition As in the real case, we denote the Weil operator of the referencing Hodge structure H by C. Then $C \in G_0$ and elements in V commute with C. Since C is a scalar multiplication on each Hodge piece, V preserves the Hermitian inner product, hence is compact.

The proof of the theorem is proceeded in steps. The verification of each step is routine, so we omitted most of them.

(1) The complexification G of G_0 is a semisimple complex Lie group. The Lie algebra $\mathfrak{g}$ of G is a real Hodge structure, polarized by the Killing form $-\mathrm{Tr}(XY)$, and whose Weil operator is $\mathrm{Ad}(C)$.

(2) Denote B by the parabolic subgroup of G_0 preserving the Hodge flag $F^\bullet(H)$, then $D^\vee = G_0/B$ is a complex projective manifold. In this case the inclusion $D \to D^\vee$ is an open immersion.

(3) The action of B on $G \times \mathfrak{g}$ defined by

$$(g, X) \mapsto (gh, \mathrm{Ad}(h)^{-1}X)$$

descends the trivial bundle $G \times \mathfrak{g}$ to a trivial bundle $\mathscr{F}$ on D, and descends to trivial bundles $F^p\mathfrak{g}$ to possibly nontrivial subbundles $\mathscr{F}^p$ of $\mathscr{F} = \mathscr{F}^{-\infty}$.

(4) Since D is a homogeneous G_0-space, the infinitesimal left translation defines a mapping of vector bundles

$$\varphi : \mathscr{F} \to T(D), \quad X \mapsto X^+.$$

Here X^+ is defined by

$$X^+_{gV}(f) = \frac{df(\exp(tX)gV)}{dt}\Big|_{t=0}.$$

Inspection at the origin shows that φ induces an isomorphism

$$\mathscr{F}/\mathscr{F}^0 \to T(D)$$

and an identification

$$\mathscr{F}^{-1}/\mathscr{F}^0 \to T^h(D).$$

(5) Letting G acting on $\mathscr{F}$ by the adjoint action, then mapping φ is equivalent with respect to the action of G.

(6) Consider the Hodge-Killing metric on $\mathscr{F}$ defined by

$$(X, Y) \mapsto -\mathrm{Tr}(\mathrm{Ad}(gCg^{-1})(X)\overline{Y})$$

at the fiber $\mathscr{F}_{gV}$, then one shows that the Hodge-Killing metric is invariant under the action of G_0. Therefore, it transports to the G_0-invariant metric under the isomorphism φ.

(7) Let X be a horizontal tangent vector at the referencing point eV of D. We need to prove that

$$\langle \Theta(X,\overline{X})X,X\rangle < 0.$$

But because of the horizontality of X, the bundle $\mathscr{F}^p$ defines a variation of Hodge structure on any integral curve of X, we can appeal the formula (†). But there one can perform an explicit computation, utilizing $\nabla_X(Y) = [X,Y]$, using that $[X,X] = 0$, we see the eventual result is $-\langle[\overline{X},X],[\overline{X},X]\rangle < 0$.

Let me justify the fact on connection used in point (7). Say $X \in \mathfrak{g}^{-1,1}$ is a horizontal tangent vector, then $X\langle Y,Z\rangle$ may be calculated as the derivative of

$$\mathrm{Tr}(\exp(tX)C\exp(-tX)Y\exp(-tX)C^{-1}\exp(tX)\overline{Z},$$

which equals

$$\mathrm{Tr}(C[X,Y]C^{-1}\overline{Z}) + \mathrm{Tr}(CYC^{-1}\overline{[\overline{X},Z]}).$$

Thus we must have $\theta_X(Y) = [X,Y]$.