DIMENSION OF VARIETIES

ALGEBRAIC GEOMETRY I (FALL 2024)

0.1. Throughout these notes, k is an algebraically closed field.

1. Rational function field

1.1. Lemma. *Let X be an irreducible variety. Then, for any open affine subset $U \subset X$, $\mathscr{O}_X(U)$ is an integral domain. Moreover, for any two open affine subsets U and U' of X, the fields of fractions of $\mathscr{O}_X(U)$ and $\mathscr{O}_X(U')$ are equal.*

Proof. Omitted. □

1.2. Definition (Rational function field). Let X be an irreducible variety over k. We define the *rational function field* of X, denoted $k(X)$, to be the field of fractions of $\mathscr{O}_X(U)$, where U is any open affine subset of X.

1.3. Lemma. *Suppose X and Y are irreducible varieties. Let $\rho\colon k(Y) \to k(X)$ be an isomorphism (as k-algebras). Then, there exist a nonempty open affine subset U of X, a nonempty open affine subset V of Y, and an isomorphism $\varphi\colon U \to V$ such that the following diagram is commutative:*

$$\begin{array}{ccc} k[V] & \xrightarrow{\varphi^\sharp} & k[U] \\ \downarrow & & \downarrow \\ k(Y) & \xrightarrow{\rho} & k(X) \end{array}.$$

Proof. Omitted. □

1.4. Definition (Birational varieties). Two irreducible varieties X and Y are said to be *birational*, if $k(X)$ is isomorphic to $k(Y)$ (as k-algebras).

A morphism $\varphi\colon X \to Y$ between irreducible varieties is said to be a *birational morphism*, if $\varphi^\sharp$ induces an isomorphism between $k(Y)$ and $k(X)$. Equivalently, φ is a birational morphism if and only if there exists an open subset U of X, such that $\varphi(U)$ is also open in Y, and $\varphi|_U$ induces an isomorphism between U and $\varphi(U)$.

1.5. Example. The projective n-space $\mathbb{P}^n$ is birational to the n-fold product $(\mathbb{P}^1)^n$. Indeed, both contain $\mathbb{A}^n$ as an open dense subset.

1.6. Example: Blowing up a point. Consider the "incidence correspondence"

$$\begin{aligned} X &= \{([Z_0, \ldots, Z_n], (T_0, \ldots, T_n)) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : T_j Z_i = T_i Z_j, 0 \leqslant i < j \leqslant n\} \\ &= \{([\ell], t) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : t \in \ell\}. \end{aligned}$$

Let $b\colon X \to \mathbb{A}^{n+1}$ be the projection to the second factor. Then b is a birational morphism. Indeed, b is an isomorphism away from $(0, \ldots, 0) \in \mathbb{A}^{n+1}$.

2. Dimension

2.1. Definition (Dimension). Let X be an irreducible variety over k. Then, the *dimension* of X is defined as the transcendence degree of $k(X)$ over k. For a review of this notion, see *Stacks Project*, `Tag 030D`.

If X is a reducible variety, then we define the dimension of X to be the maximum of the dimensions of the irreducible components of X.

2.2. A variety of dimension one is called a *curve*. A variety of dimension two is called a *surface*. An n-dimensional variety is called an *n-fold*.

2.3. Examples. Two birational irreducible varieties have the same dimension. If X is irreducible and n-dimensional, then any nonempty open subvariety of X is also n-dimensional.

The dimension of $\mathbb{A}^n$ is n. The dimension of $\mathbb{P}^n$ is n.

2.4. Lemma. *Let X be an irreducible variety, and Y a proper closed subvariety of X. Then* $\dim Y < \dim X$.

Proof. It suffices to prove the statement under the assumption that X is affine and Y is irreducible. In this case, $A := k[X]$ is an integral domain, and $k[Y] = A/\mathfrak{p}$ for some nonzero prime ideal $\mathfrak{p}$ of A. The lemma is then equivalent to demonstrating that the transcendence degree of A is strictly greater than that of $A/\mathfrak{p}$. (We adopt the convention that the transcendence degree of an integral domain over k is defined as that of its field of fractions.)

Suppose the transcendence degree of A over k is n. If the statement is false, then we can find elements $x_1, \dots, x_n$ in A that are algebraically independent, such that their images $\overline{x}_1, \dots, \overline{x}_n$ in $A/\mathfrak{p}$ remain algebraically independent. Next, choose an element $a \in \mathfrak{p}$ with $a \neq 0$. Since the transcendence degree of A is n, the $n+1$ elements $a, x_1, \dots, x_n$ must be algebraically dependent. Therefore, we can find an irreducible polynomial $f(T_0, \dots, T_n) \in k[T_0, \dots, T_n]$ such that

$$f(a, x_1, \dots, x_n) = 0.$$

Since $a \neq 0$, it follows that f is not a polynomial in T_0. In other words, $f(0, T_1, \dots, T_n)$ is not the zero polynomial. Consequently, the equation above, when considered modulo $\mathfrak{p}$, yields a nontrivial algebraic relation:

$$f(0, \overline{x}_1, \dots, \overline{x}_n) = 0$$

among $\overline{x}_1, \dots, \overline{x}_n$. This leads to a contradiction. □

2.5. Definition (Dominant morphism). A morphism $\varphi\colon X \to Y$ is said to be a *dominant morphism*, if $\varphi(X)$ is dense in Y.

2.6. Lemma. *Let $\varphi\colon X \to Y$ be a morphism between irreducible affine varieties. Then φ is dominant if and only if $\varphi^\sharp\colon k[Y] \to k[X]$ is injective.*

Proof. Omitted. □

2.7. Lemma. *Let $\varphi\colon X \to Y$ be a dominant morphism. Then* $\dim X \geqslant \dim Y$.

Proof. Omitted. □

3. Finite morphism and dimension

3.1. A morphism $\varphi\colon X \to Y$ is said to be a *finite morphism*, if

- for any open affine subset U of Y, $\varphi^{-1}(U)$ is also affine,
- for any open affine subset U of Y, the k-algebra homomorphism $\varphi_U^\sharp\colon k[U] \to k[\varphi^{-1}(U)]$ is integral. That is, for any $g \in k[\varphi^{-1}(U)]$, there exist an integer $d \geqslant 0$, and $a_0, \dots, a_{d-1} \in k[U]$, such that

$$g^d + a_{d-1}g^{d-1} + \cdots + a_1 g + a_0 = 0 \quad \text{in } k[\varphi^{-1}(U)].$$

3.2. Lemma. *Suppose $\varphi\colon X \to Y$ is a finite morphism between varieties.*

(1) *If φ is dominant, then φ is surjective,*
(2) *The morphism φ is a closed map.*

Proof. The problem being local, we can assume X and Y are affine.

Proof of (1). Since φ is dominant, $\varphi^\sharp\colon k[Y] \to k[X]$ is injective (2.6). Since $\varphi^\sharp$ is an integral extension, the "Going-up Theorem" of Cohen and Seidenberg (*Stacks Project*, `Tag 00GQ`) implies that any prime ideal of $k[Y]$ is of the form $(\varphi^\sharp)^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is some prime ideal of $k[X]$. Using the ideal-variety correspondence, this implies that φ is surjective.

Let us now turn to (2), still assuming X and Y are affine. Let Z be an irreducible closed subset of X. Let W be the closure of $\varphi(Z)$ in Y. Then one checks (exercise) that $\varphi|_Z\colon Z \to W$ is still a finite morphism and is dominant by fiat. Applying the discussion from the previous paragraph, we see $\varphi(Z) = W$, hence it is closed in Y. □

One can construct many finite morphisms on an affine variety.

3.3. Theorem (Noether normalization)**.** *Let X be an irreducible affine variety. Then there exists a finite dominant morphism $\pi\colon X \to \mathbb{A}^n$ for some n.*

Proof. See *Stacks Project*, `Tag 00OY`. □

3.4. Lemma. *Let $\varphi\colon X \to Y$ be a finite dominant morphism between varieties. Then X and Y have the same dimension.*

Proof. First, let us assume that both X and Y are irreducible. Since the involved varieties are irreducible, the dimension is unchanged after passing to an open subset. Thus, we can further assume X and Y are affine. In this case, $\varphi^\sharp\colon k[Y] \to k[X]$ is an injective k-algebra homomorphism (2.6) and is an integral extension. Thus, any $f \in k[X]$ is in particular algebraic over $k[Y]$. This implies that $k(X)$ is an algebraic extension of $k(Y)$ (as extensions of k). Therefore, $k(X)$ and $k(Y)$ have the same transcendence degree, hence the dimensions of X and Y are equal.

If Y is irreducible but X is arbitrary, then we can write X as a union of its irreducible components $X = \bigcup X_e$. For each e, $\varphi|_{X_e}\colon X_e \to Y$ is a finite morphism. If $\varphi(X_e)$ is dense in Y, then we have $\dim X_e = \dim Y$ by the previous paragraph. Otherwise, $\varphi(X_e)$ has strictly smaller dimension (2.4), and X_e is finite dominant over a variety of dimension less than $\dim X$. Thus, $\dim X_e < \dim X$. Therefore, we have shown $\dim X \leqslant \dim Y$. But since $\bigcup_e \varphi(X_e) = X$, there must exist some e such that X_e dominates Y. So the equality must hold.

Now suppose Y is reducible, and write Y as the union of its irreducible components $Y = \bigcup Y_e$. Since $X_e := \varphi^{-1}(Y_e)$ is finite dominant over Y_e, we have $\dim X_e = \dim Y_e$. Since $X = \bigcup X_e$, each irreducible component must be some irreducible component of X_e. This suffices to imply $\dim X = \dim Y$. □

3.5. Lemma. *Let $\varphi\colon X \to Y$ be a dominant morphism of varieties. Then $\varphi(X)$ contains a nonempty open subset of Y.*

Proof. We can assume that X and Y are irreducible and affine. Then, $\varphi^\sharp\colon k[Y] \to k[X]$ is injective (2.6), and it induces a field extension $\varphi^*\colon k(Y) \to k(X)$. This extension being finitely generated, we can find $T_1, \dots, T_r \in k(X)$ such that $k(X)$ is a finite extension of $k(Y)(T_1, \dots, T_r)$. After replacing X with a smaller open subset, we can assume $T_i \in k[X]$. Thus, we get an injective ring homomorphism

$$k[Y \times \mathbb{A}^r] \to k[X]$$

which induces the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varpi} & Y \times \mathbb{A}^r \\ & \underset{\varphi}{\searrow} & \downarrow \mathrm{pr}_2 \\ & & Y \end{array}$$

The morphism ϖ is nearly finite. In fact, every element $f \in k[X]$ satisfies a polynomial relation

$$a_d f^d + \cdots + a_1 f + a_0 = 0, \quad a_i \in k[Y \times \mathbb{A}^r],$$

but a_d is generally not a unit. Nevertheless, we can do the following to remedy this. Since $k[X]$ is of finite type over $k[Y \times \mathbb{A}^r]$, we can find $f_1, \dots, f_s \in k[X]$ that generate $k[X]$ as a $k[Y \times \mathbb{A}^r]$-algebra. Suppose f_j satisfies a polynomial relation

$$b_j f_j^{d_j} + \cdots = 0, \quad (b_j \in k[Y \times \mathbb{A}^r]).$$

Then we define $G = b_1 \cdots b_s \in k[Y \times \mathbb{A}^r]$, $U = \{(y,t) \in Y \times \mathbb{A}^r : G(y,t) \neq 0\}$, and $V = \{x \in X : G(\varpi(x)) \neq 0\}$. By construction, ϖ induces a finite dominant morphism $V \to U$. By (3.2), $\varpi(V) = U$, and we have constructed a nonempty open subset U contained in $\varphi(X)$. □

3.6. Let us summarize the construction used in the proof of (3.5). For any morphism $\varphi\colon X \to Y$ between irreducible affine varieties, we can find:

- an open affine immersion $j\colon X_1 \to X$;
- a morphism $\varpi\colon X_1 \to Y \times \mathbb{A}^r$ such that $\mathrm{pr}_2 \circ \varpi = \varphi \circ j$;
- a principal open subset U of $Y \times \mathbb{A}^r$ such that ϖ induces a finite morphism $\varpi^{-1}(U) \to U$.

In other words, we have the following commutative diagram:

$$\begin{array}{ccccc} U & \longleftarrow & \varpi^{-1}(U) & & \\ \downarrow & & \downarrow & & \\ Y \times \mathbb{A}^r & \xleftarrow{\varpi} & X_1 & \xrightarrow{j} & X \\ & \underset{\mathrm{pr}_1}{\searrow} & & \underset{\varphi}{\swarrow} & \\ & & Y & & \end{array}.$$

In this diagram, the hooked arrows are open immersions, and the upper horizontal arrow is finite.

Here are two notions which are closely related to finite morphisms.

3.7. Definition (Quasi-finite morphism). A morphism $\varphi\colon X \to Y$ between varieties is *quasi-finite*, for any $y \in Y$, $\varphi^{-1}(y)$ is a finite set.

3.8. Example. Any finite morphism is quasi-finite. If $\varphi\colon X \to Y$ is finite, and $U \subset X$ is a locally closed subset, then $\varphi|_U$ is quasi-finite.

3.9. Vista. One version of Zariski's main theorem states that if $\varphi\colon X \to Y$ is quasi-finite, then there is a commutative diagram

$$\begin{array}{ccc} X & \xhookrightarrow{\ j\ } & X' \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi'} \\ & & Y \end{array}$$

in which j is an open immersion, and φ' is a finite morphism. We will not use this deep theorem.

3.10. Definition (Generically finite morphism). A morphism $\varphi\colon X \to Y$ between varieties is *generically finite*, if there is an open subset U of Y, such that $\varphi|_{\varphi^{-1}(U)}\colon \varphi^{-1}(U) \to U$ is quasi-finite.

3.11. Lemma. *Let $\varphi\colon X \to Y$ be a generically finite morphism between irreducible varieties. Then* $\dim X = \dim Y$.

Proof. Without loss of generality, we can assume φ is already quasi-finite. Let us use the construction of (3.6). Then $\varphi \circ j = \mathrm{pr}_1 \circ \varpi$ is quasi-finite. If $r > 0$, then $\mathrm{pr}_1^{-1}(y)$ is infinite for any $y \in Y$. Therefore, there exists $y \in Y$ such that $\mathrm{pr}_2^{-1}(y) \cap U$ is dense in $\mathrm{pr}_1^{-1}(y)$, whence finite. This implies that $\varpi^{-1}(\mathrm{pr}_1^{-1}(y) \cap U)$ is infinite. In particular, $(\mathrm{pr}_1^{-1} \circ \varpi^{-1})(y)$ is infinite, a contradiction. The contradiction shows that we must have $r = 0$. In this case, we have $\dim X = \dim X_1 = \dim \varpi^{-1}(U) = \dim U = \dim Y$, as desired. □

3.12. Corollary. *Let $\varphi\colon X \to Y$ be a quasi-finite, dominant morphism between (possibly reducible) varieties. Then* $\dim X = \dim Y$.

4. Hauptidealsatz

The goal of this section is to prove the following theorem.

4.1. Theorem (Krull's Hauptidealsatz)**.** *Let X be an irreducible variety, and $f \in \mathscr{O}_X(X)$. Suppose that f is not identically zero, and f is not a unit in $\mathscr{O}_X(X)$. Let V be an irreducible component of $\{x \in X : f(x) = 0\}$. Then*

$$\dim V = \dim X - 1.$$

4.2. The theorem holds for $X = \mathbb{A}^n$.

It suffices to assume f is an irreducible polynomial, so $\{f = 0\}$ is irreducible as a variety. Then a generic projection from a point not in X exhibits a quasi-finite, dominant morphism from X to $\mathbb{A}^{n-1}$. We can now apply 3.11.

4.3. It suffices to prove the theorem under the hypothesis that X is affine. From now on, we enforce this hypothesis.

4.4. It suffices to prove the theorem under the hypothesis that $\{f = 0\}$ is irreducible.

To see this, write $\{f = 0\} = W_1 \cup W_2 \cup \cdots \cup W_r$, where W_i are the irreducible components of $\{f = 0\}$. If we want to prove that W_1 satisfies $\dim W_1 = \dim X - 1$, then we can choose an open affine subset X' of X such that X' is disjoint from $W_2, \dots, W_r$, while $W_1 \cap X'$ is nonempty. Since W_1 is irreducible, $X' \cap W_1$ is also irreducible, and W_1 is the only irreducible component of $\{f|_{X'} = 0\} \subset X'$. The theorem for X' then implies $\dim(W_1 \cap X') = \dim X' - 1$, which gives the desired result.

From now on, we shall enforce the hypothesis that $V \coloneqq \{f = 0\}$ is irreducible.

4.5. Construction. Assume that $\dim X = n$. Choose a finite dominant morphism $\pi\colon X \to \mathbb{A}^n$. Using $\pi^\sharp$, we identify $k[T_1, \dots, T_n]$ as a subring of $k[X]$. Thus, $f \in k[X]$ satisfies a minimal integral relation

$$f^e + a_{e-1} f^{e-1} + \cdots + a_1 f + a_0 = 0, \quad a_i \in k[T_1, \dots, T_n],\ a_0 \neq 0.$$

Consider the morphism

$$\pi \times f\colon X \to \mathbb{A}^n \times \mathbb{A}^1.$$

Then the image of $\pi \times f$ lands in the hypersurface

$$Y = \{(t_1, \dots, t_n, z) : z^e + a_{e-1}(t) z^{e-1} + \cdots + a_0(t) = 0\}.$$

We thus get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varpi\ } & Y \\ & \searrow^{\pi} & \downarrow{\scriptstyle p} \\ & & \mathbb{A}^n \end{array}$$

where ϖ is the restriction of $\pi \times f$ and p is the restriction of the projection. It is clear that all three morphisms in the diagram are finite morphisms. Since $k[Y] = k[\mathbb{A}^n][f] \subset k[X]$, it is necessarily an integral domain, hence irreducible. Thus, ϖ is a finite surjective morphism (3.2).

4.6. Proof of Theorem 4.1. Note that $V = \varpi^{-1}(Y \cap (\mathbb{A}^n \times \{0\}))$. Since $Y \cap (\mathbb{A}^n \times \{0\})$ is the hypersurface in $\mathbb{A}^n$ defined by $\{a_0(t) = 0\}$, by (4.2) again, we see that the dimension of each irreducible component of $Y \cap (\mathbb{A}^n \times \{0\})$ is $n-1$. However, ϖ is surjective, and, since V is irreducible, $\varpi(V)$ must be equal to $Y \cap (\mathbb{A}^n \times \{0\})$. By (3.4), we see $\dim V = n-1$. This completes the proof. □

Here are some immediate applications of the Hauptidealsatz.

4.7. Corollary. *Let X be an irreducible variety. Let Z be a maximal irreducible closed subset of X not equal to X. Then* $\dim Z = \dim X - 1$.

4.8. Corollary. *Let X be a variety. Then the dimension of X is equal to the largest integer $d \geqslant 0$, such that there exists a chain of irreducible closed subsets*

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d.$$

In other words, the dimension we defined using transcendence degree agrees with the "Krull dimension" defined in *Stacks Project*, Tag 0055.

4.9. Corollary. *Let X be an irreducible variety. Let $f_1, \dots, f_r \in \mathscr{O}_X(X)$ be regular functions. Assume that $\{f_1 = \cdots = f_r = 0\}$ is nonempty. Let V be an irreducible component of $\{f_1 = \cdots = f_r = 0\}$. Then* $\dim V \geqslant \dim X - r$.

4.10. Corollary. *Let $\varphi\colon X \to Y$ be a morphism of irreducible varieties. Assume that* $\dim X - \dim Y = r \geqslant 0$. *Then for any $x \in X$,* $\dim \varphi^{-1}(\varphi(x)) \geqslant r$.

Proof. Using the Noether normalization, we can assume $Y = \mathbb{A}^r$. Then we can apply the previous corollary. □

5. Chevalley's theorem

5.1. Lemma. *Let $\varphi\colon X \to Y$ be a dominant morphism of irreducible varieties. Assume that $r =$* $\dim X - \dim Y$. *Then there is a Zariski open dense subset U of Y satisfying the following properties:*

- *$U \subset \varphi(X)$,*
- *for any $y \in U$ and any irreducible component V of $\varphi^{-1}(y)$, we have* $\dim V = r$.

Proof. It is clear that we can assume Y is affine. If $X = \bigcup U_i$ is an open covering of X, then the second assertion is equivalent to $\dim V \cap U_i = r$ if the intersection is nonempty. So we can assume X is affine as well.

We shall prove the lemma by induction on the dimension of X. With the hypothesis that X and Y are affine, we apply again the construction in (3.6). We add a remark that $\mathrm{pr}_1 : Y \times \mathbb{A}^r \to Y$ is an open map, which is easily checked. Let $W_1 \cup \cdots \cup W_r$ be the irreducible component decomposition of $X \setminus X_1$. Then we have $\dim W_i < \dim X$. Therefore, there is an open subset U_1 of Y, such that for any $y \in U_1$, $\dim \varphi^{-1}(y) \cap W_i < r$.

Without loss of generality, we can assume the open set U provided by (3.6) satisfies that $\mathrm{pr}_1(U) \subset U_1$. Then, for each $y \in \mathrm{pr}_1(U)$, we already know from (4.10) that if V is an irreducible component of $\varphi^{-1}(y)$, we have $\dim V \geqslant r$. Therefore, V cannot be entirely contained in $X \setminus X_1$ by the discussion of the previous paragraph. Hence $V \cap X_1$ is nonempty, and it suffices to show that any irreducible component of $\varphi^{-1}(y) \cap X_1$ has dimension r.

For the chosen y, we know $U_y = U \cap \{y\} \times \mathbb{A}^r$ is r-dimensional. Since ϖ is surjective, $\varpi^{-1}(y) \cap X_1 = \varpi^{-1}(U_y)$ is finite and dominant over U_y. Thus, any irreducible component of $\varpi^{-1}(y) \cap X_1$ has dimension $\leqslant r$. In view of (4.10), we see equality must hold. This completes the proof. □

5.2. Theorem (Chevalley)**.** *Let $\varphi\colon X \to Y$ be a morphism of varieties. Then the function*

$$x \mapsto \dim \varphi^{-1}(\varphi(x))$$

is upper semicontinuous. In other words, for each natural number m,

$$\{x \in X : \dim \varphi^{-1}(\varphi(x)) \geqslant m\}$$

is closed in X.

Proof. First reduce to the irreducible case, then use induction on dimension, aided by (5.1). Details are omitted. □

5.3. Corollary. *Let $\varphi\colon X \to Y$ be a morphism of varieties. Assume that φ is a closed map. Then the function*

$$y \mapsto \dim \varphi^{-1}(y)$$

is upper semicontinuous. In other words, for each natural number m,

$$\{y \in Y : \dim \varphi^{-1}(y) \geqslant m\}$$

is closed in Y.

Proof. This is because φ is a closed map, and that $\varphi\{x \in X : \dim \varphi^{-1}(\varphi(x)) \geqslant m\} = \{y \in Y : \dim \varphi^{-1}(y) \geqslant m\}$. □

5.4. Corollary. *Let $\varphi\colon X \to Y$ be a surjective closed morphism between varieties. Suppose that Y is irreducible, and there exists an integer r, such that for any $y \in Y$, $\varphi^{-1}(y)$ is irreducible of dimension r. Then X is irreducible of dimension* $\dim Y + r$.

Proof. Exercise. □

5.5. Lines on surfaces. Let $d \geqslant 1$. Let $\mathbb{P}^N$ be the parameter space of all degree d hypersurfaces in $\mathbb{P}^3$. Let us consider the incidence correspondence

$$\mathscr{X} = \{(L, S) \in \mathbb{G}(1,3) \times \mathbb{P}^N : L \subset S\}.$$

First, let us examine the first projection $p_1\colon \mathscr{X} \to \mathbb{G}(1,3)$. For a fixed line, the space of all degree d surfaces containing L is an $(N-d-1)$-dimensional linear subspace of $\mathbb{P}^N$ (indeed, it suffices to check it for a special line, since $\mathrm{GL}_4(k)$ operates transitively on $\mathbb{G}(1,3)$). Since $\mathbb{G}(1,3)$ is irreducible of dimension 4, we see $\mathscr{X}$ is irreducible of dimension $N-d+3$, by (5.4). If $d \geqslant 4$, the second projection $p_2\colon \mathscr{X} \to \mathbb{P}^N$ cannot be dominant. This shows that a general degree $d \geqslant 4$ surface in $\mathbb{P}^3$ contains no line.

If $d = 3$, either $p_2\colon \mathscr{X} \to \mathbb{P}^N$ is surjective, or its image is a proper closed subset of $\mathbb{P}^N$. The latter possibility is ruled out because it would imply that if a cubic surface contains a line, then it contains infinitely many lines. But we have seen that on the Fermat cubic surface, there are exactly 27 lines. So it must be that p_2 is surjective. This shows that every cubic surface contains at least one line.