THE DE RHAM COHOMOLOGY OF FORMAL POWER SERIES

Added on [2017-12-14 Thu]. Over the field of complex numbers, the cohomology of a variety Z can be computed in the following fashion. We embed Z into a smooth variety X, make the formal completion $\mathfrak{X}_{/Z}$ of X along Z. So $\mathfrak{X}_{/Z}$ is a formal scheme. Then one take the de Rham cohomology of $\mathfrak{X}_{/Z}$, and this cohomology is isomorphic to the cohomology of Z. [We should not use the sheaf of Kähler differentials directly, but should use completed Kähler differential, to compute de Rham cohomology. For instance, when X is affine, Z defined by an ideal I, the completed Kähler differential is the I-adic completion of the usual Kähler differential]

For Z smooth, this assertion is proven by Grothendieck. I don't know the reference for the proof of this general assertion; but one can consult a crystalline paper by Bhatt and de Jong.

The (wrong) analogue of this assertion in the p-adic setting is the following. If we have a variety Z over $\mathbb{F}_p$, we expect, ideally, that we should be able to compute its cohomology using the formal completion of a smooth scheme over $\mathbb{Z}_p$ along Z. But this post shows that this is not the case: we can embed a point it into $\mathbb{A}^1_{\mathbb{Z}_p}$, then the completion of $\mathbb{Z}_p[t]$ along the ideal (p,t) is $\mathbb{Z}_p[\![t]\!]$, whose (completed) de Rham cohomology is *not* the same as any desired cohomology of a point (it is too large, even after killing torsions).

0.1. Notations.

- Let K be a nonarchimedean field with valuation ring $\mathfrak{o}_K$ and maximal ideal $\mathfrak{m}_K$.
- Let $\varpi \in \mathfrak{m}_K$ be a pseudo-uniformizer of K.
- Let $k = \mathfrak{o}_K/\mathfrak{m}_K$ be the residue field of K.
- Assume k has characteristic $p > 0$.
- Equip $\mathfrak{o}_K[\![T]\!] = \mathfrak{o}_K[\![T_1, \dots, T_n]\!]$ with the (ϖ, T)-adic topology.

1. The de Rham cohomology of power series modulo torsion

1.1. *The de Rham cohomology of the polynomial ring on $\mathfrak{o}_K$ is ϖ-power torsion.* Here is the indication of the proof of this assertion. For simplicity let us consider the case $n = 1$.

- Then an integral of a differential form $\eta = \sum_{n=1}^d a_n T^n \frac{\mathrm{d}T}{T}$ in $K[T]$ is given by $\sum_{n=1}^d \frac{a_n}{n} T^n$. Take m sufficiently large such that $|\varpi^m/n| \leq 1$, $\varpi^m\eta$ is exact, and the de Rham cohomology class of η is ϖ^m-torsion.

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⋄ Thus $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{o}_K[T]/\mathfrak{o}_K)[\varpi^{-1}] = 0$ for all $i > 1$.
What about the de Rham cohomology of $\mathfrak{o}_K[\![T]\!]$?

1.2. Remark on rigid geometry. Berthelot's "generic fiber" of $\mathrm{Spf}(\mathfrak{o}_K[\![T]\!])$ is precisely the "tubular neighborhood" of $]\{0\}[_{\mathrm{Sp}(K\langle T\rangle)}$ inside the rigid analytic closed unit disk — which is the *open* unit disk. This is better explained in terms of Huber's adic space: the generic fiber of the natural mapping $\Phi : \mathrm{Spa}(\mathfrak{o}_K[\![T]\!], \mathfrak{o}_K[\![T]\!]) \to \mathrm{Spa}(\mathfrak{o}_K, \mathfrak{o}_K)$ **is** the rigid analytic open disk. See this post.

1.3. Remark on rigid cohomology. Recall that the de Rham cohomology of the *open* unit disk **is** the correct cohomology of the zero dimensional scheme 0: the rigid cohomology (at least for smooth proper schemes) is defined by the de Rham cohomology of their tubular neighborhoods inside some liftable scheme.

1.4. One is tempted to believe (although being *wrong*) that the cohomology of the algebraic de Rham complex $\Omega^\bullet_{\mathfrak{o}_K[\![T]\!]/\mathfrak{o}_K}$ should "reflect" the cohomology of a "point", viewed as the origin point of the affine space $\mathbb{A}^n_k$. For instance, one is tempted to believe the de Rham cohomology of $\mathfrak{o}_K[\![T]\!]$ is zero after inverting ϖ. But *this is wrong.*

1.5. *The de Rham cohomology of* $\mathfrak{o}_K[\![T]\!]$ *is infinitely generated over* $\mathfrak{o}_K$, and *inverting* ϖ *will not kill it.* The thrust is that $\mathfrak{o}_K[\![T]\!][1/\varpi]$ is equal to the ring of *bounded functions* on the open unit disk; it is not large enough to contain all integrals of functions.

1.6. Example. Let us consider $\eta = \sum_{n=1}^{\infty} T^{p^n} \frac{\mathrm{d}T}{T}$. One integral of η is $f(T) = \sum T^{p^n}/p^n$. No matter how large m is, $p^m f(T)$ will not fall into $\mathfrak{o}_K[\![T]\!]$. Any constant multiple of an unbounded function is still unbounded.

1.7. In fact the torsion free part of the de Rham cohomology of $\mathfrak{o}_K[\![T]\!]$, even when the number of the variable equals one, is large enough to contain all the coherent cohomology of most of algebraic curves (with varying genera). Below we only consider the case of an elliptic curve. But the spirit should be clear to be applicable for curves of higher genus.

2. Formal expansion of closed differentials on an elliptic curve.

2.1. Assume that $p > 2$. Let $E/\mathfrak{o}_K$ be an elliptic curve with an *ordinary good reduction*, expressed as the projective completion of a Legendre curve. Let $\eta = \mathrm{d}x/y$ be the differential 1-form that generates $\mathrm{H}^0(E, \Omega^1_{E/\mathfrak{o}_K})$. Let $P = [0, 0, 1]$ be an $\mathfrak{o}_K$-point on E. Then we can expand η in a formal neighborhood of P. This formal neighborhood, up on choosing a coordinate function near P, is isomorphic to $\mathfrak{o}_K[\![T]\!]$. Since η is a closed form, it formal expansion defines a de Rham cohomology class in $\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{o}_K[\![T]\!]/\mathfrak{o}_K)$.

2.2. Claim. *The "formal expansion" of* η *defines a nonzero de Rham cohomology class.*

2.3. Proof of Claim 2. It is without loss of generality to assume that k is algebraically closed, so I will not worry about the meaning of "points". Also I assume that $(p) = \mathfrak{m}_K$, purely for convenience.

- ◇ Write $\eta = \sum_{n\geq 1} a_n T^n \frac{\mathrm{d}T}{T}$ with $a_n \in \mathfrak{o}_K$, $a_1 = 1$. Then being formally exact means $a_n \in n\mathfrak{o}_K$ for all $n \geq 1$. Recall that E is assumed to be ordinary, then the claim follows at once if we can show a_p agrees with the *Hasse invariant* of E modulo p (if we use η as the basis to define the Hasse invariant). So p must not divide a_p and η is not exact.
- ◇ To prove the assertion we use Serre duality, which asserts that, if $E_k = E \otimes_{\mathfrak{o}_K} k$, then there is a canonical perfect pairing
$$\mathrm{H}^1(E_k, \mathcal{O}_{E_k}) \otimes \mathrm{H}^0(E_k, \Omega^1_{E_k/k}) \to k.$$
Thus Hasse invariant under a basis $\mathbf{f}$ is equal to the ratio $\langle F^*_{\mathrm{abs}}(\mathbf{f}), \eta\rangle / \langle \mathbf{f}, \eta\rangle$.
- ◇ To this end, it is more convenient to use the classical adèlic language. Let us explain how to represent an element in the group $\mathrm{H}^1(E_k, \mathcal{O}_{E_k})$ in this context. A *répartition*, or an *adèle* on E_k is an element
$$\mathbf{f} = (f_x) \in \prod_{x\in|E_k|} \mathrm{Frac}(\widehat{\mathcal{O}}_{E_k,x})$$
such that $f_x \in \widehat{\mathcal{O}}_{E_k,x}$ for all but finitely many $x \in |E_k|$. The ring of all adèles form a ring $\mathcal{R} = \mathcal{R}_{E_k}$. Clearly each rational function $f \in k(E_k) = \mathrm{Frac}(E_k)$ defines an adèle called a *principal adèle*. Also, the infinite product $\mathcal{R}(0) = \prod_x \widehat{\mathcal{O}}_{E_k,x}$ is a subring of $\mathcal{R}$. We then have an isomorphism
$$\mathrm{H}^1(E_k, \mathcal{O}_{E_k}) = \frac{\mathcal{R}}{\mathcal{R}(0) + k(E_k)}.$$
Moreover, the Serre duality is established, under this identification, by saying that the pairing
$$(\mathbf{f}, \eta) \mapsto \mathrm{Res}(\mathbf{f}\eta) = \sum_{x\in|E_k|} \mathrm{Res}_x(f_x\eta) : \mathcal{R}/(\mathcal{R}(0) + k(E_k)) \times \mathrm{H}^0(E_k, \Omega^1_{E_k}) \to k$$
is a perfect pairing.
- ◇ Let $\mathbf{f} = (f_x)$ be an adèle such that $f_x \in \widehat{\mathcal{O}}_{E_k,x}$ for $x \neq o$, and f_o has a single pole of order one. Thus the class of $\mathbf{f}$ form a basis of $\mathrm{H}^1(E_k, \mathcal{O}_{E_k})$ (as $\mathrm{h}^1(\mathcal{O}_{E_k})$ is 1-dimensional).
- ◇ Next we consider the Frobenius action. Since the Frobenius action on adèles is compatible with the Frobenius on functions, the class of the adèle $\mathbf{f}^p$ represents $F^*_{\mathrm{abs}}[\mathbf{f}]$. So we can write $[\mathbf{f}^p] = c \cdot [\mathbf{f}]$, for some $c \in k$, then c **is** the *Hasse invariant* of E_k with respect to the basis $[\mathbf{f}]$. Recall that our hypothesis is that $c \neq 0$.
- ◇ In order to relate c with the differential form, we consider $\mathrm{Res}(\mathbf{f}\eta) = \mathrm{Res}_o(f_o^p\eta)$. By definition, this is the coefficient of the T^{-1} term of the rational differential $f_o^p\eta$ under the formal expansion. Since f_o has a pole of order 1 at o, and since the

residue of $f_o\eta$ at o equals 1, we conclude that the residue of $f_o^p\eta$ equals $a_p \pmod p$, the coefficient of $T^{p-1}\mathrm{d}T$ term in the formal expansion modulo p. Since the Serre duality paring is functorial, it follows that we must have $a_p \equiv c \pmod p$. This completes the proof.

2.4. Since $\mathrm{H}^0(E, \Omega^1_{E/\mathfrak{o}_K})$ is a free $\mathfrak{o}_K$-module of rank 1, the above also shows that the $\mathrm{H}^1(E, \Omega^1_{E/\mathfrak{o}_K})$ injects into the torsion free part of the formal de Rham group via the formal expansion. This justifies (1.5).

2.5. Frobenius action on formal power series. In the rest of these notes we will be dealing with crystalline cohomology of E. We want to show that $\mathrm{H}^0(E, \Omega^1_{E/\mathfrak{o}_K})$ is a *complement* to the "slope zero part" of the de Rham cohomology.

This is to serve as an instance of the "Newton-Hodge decomposition theorem" of Katz, an algebraic theorem proved for "F-crystals" (Katz, N. M. (1979). *Slope filtration of F-crystals.* In "Journées de Géométrie Algébrique de Rennes" (Rennes, 1978), Vol. I (pp. 113–163). Soc. Math. France, Paris.) In the presence of geometry this could be understood fairly nicely (see Katz, N. (1973). *Travaux de Dwork.* Séminare Bourbaki). Unfortunately, the language we employed below is too fancy compared to its elementary nature, and is fancier than it should be.

- ◇ Let $\phi : \mathfrak{o}_K[\![T]\!] \to \mathfrak{o}_K[\![T]\!]$ be a Frobenius lift on $\mathfrak{o}_K[\![T]\!]$. For instance, if $\mathfrak{o}_K$ is the ring of Witt vectors of a perfect field k, then we can take

$$\phi\left(\sum_{n=0}^{\infty} a_n T^n\right) = \sum_{n=0}^{\infty} \sigma(a_n) T^{pn}$$

  where σ is the canonical lift of Frobenius. Or, if we do not require the Frobenius lift to be absolute, we could just take

$$\phi\left(\sum_{n=0}^{\infty} a_n T^n\right) = \sum_{n=0}^{\infty} a_n T^{pn}$$

- ◇ The point is that either of the above Frobenii together with the exterior differential d, defines a "(Φ, ∇)-module structure" on $\mathfrak{o}_K[\![T]\!]$. That is, the diagram

$$\begin{array}{ccc} \mathfrak{o}_K[\![T]\!] & \xrightarrow{\ \phi\ } & \mathfrak{o}_K[\![T]\!] \\ {\scriptstyle \mathrm{d}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}} \\ \mathfrak{o}_K[\![T]\!]\mathrm{d}T & \xrightarrow[\phi\otimes \mathrm{d}\phi]{} & \mathfrak{o}_K[\![T]\!]\mathrm{d}T \end{array}$$

  is commutative. It follows that there is an action of the Frobenius on $\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{o}_K[\![T]\!]/\mathfrak{o}_K)$, the cokernel of d.

- ◇ Now assume that K is unramified, then the identification between the crystalline cohomology and de Rham cohomology gives rise to also a Frobenius structure on

$\mathrm{H}^1_{\mathrm{dR}}(E/\mathfrak{o}_K)$. The restriction map, obtained from the "formal expansion",

$$\mathfrak{e} : \mathrm{H}^1_{\mathrm{dR}}(E/\mathfrak{o}_K) \to \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{o}_K[\![T]\!]/\mathfrak{o}_K)$$

respect the actions of Frobenii on both sides.

- ◇ **Note.** the functoriality of Frobenii actions is highly nontrivial; since the Frobenius lift we chose can be arbitrary, and if E is more general than elliptic curves, it is not even clear what *is* the Frobenius action on the de Rham cohomology of E. The existence of Frobenii on cohomology, the independence of the Frobenius lifts, are all the gifts from the magic of crystalline cohomology.

2.6. Abstract Hodge slopes. Let $H = \mathrm{H}^1_{\mathrm{dR}}(E/\mathfrak{o}_K)$. Since H is a p-torsion free $\mathfrak{o}_K$-module, the Frobenius action is determined by its action on an $\mathfrak{o}_K$-basis of H. Namely, if we express $e = x_1e_1 + x_2e_2$ with respect to a basis $\{e_1, e_2\}$, then $\Phi(e) = (e_1, e_2)F(x_1, x_2)^{\mathrm{T}}$ for some matrix F. Choosing a different basis yields a different matrix representation G. If U is a change-basis matrix, then $G = UF\phi(U)^{-1}$. The classical "invariant factor theorem" for matrices over a discrete valuation ring applies; there exist $A, B \in \mathrm{GL}_2(\mathfrak{o}_K)$ such that $AFB = \mathrm{diag}(p^{i_1}, p^{i_2})$, $i_1 \le i_2$. The *abstract Hodge slopes* of H is defined to be the multiset $\{i_1, i_2\}$.

2.7. Here are the upshots:

- ◇ the isomorphism class of $\mathrm{H}^1_{\mathrm{dR}}(E/\mathfrak{o}_K) \otimes K$ is determined by a multiset known as the "Newton slopes";
- ◇ when E has an ordinary reduction, the Newton slopes agree with the abstract Hodge slopes on the p-divisibility of the Frobenius action Φ;
- ◇ in our situation (i.e., H and $\mathrm{H}^0(E, \Omega^1_{E/\mathfrak{o}_K})$ are free over $\mathfrak{o}_K$) the multiset of abstract Hodge slopes is equal to the "geometric Hodge slopes", which is $\{0, 1\}$.

In general, the "geometric Hodge slopes" for X at cohomological degree m are 0 ($h^{m,0}$ times), 1 ($h^{m-1,1}$ times), ..., m ($h^{0,m}$ times). That the geometric and abstract Hodge slopes agree, under some freeness hypotheses, is the main theorem of [B. Mazur (1972), *Frobenius and the Hodge filtration*, Bull. Amer. Math. Soc., 78, 653–667.]

2.8. Now we perform a simple computation: for a 1-form $\omega = \sum a_nT^n\frac{\mathrm{d}T}{T}$, we have

$$(\phi \otimes \mathrm{d}\phi)(\omega) = p\sum_{n\ge1} na_nT^n\frac{\mathrm{d}T}{T}.$$

It follows that $\mathfrak{e}(\Phi(a))$ *is divisible by* p.

By the ordinary assumption, we must have $\mathfrak{e}(U) = 0$, where U is the slope zero (synonym: unit-root) part of H! Now U is killed by the formal expansion, and $\mathrm{H}^0(E, \Omega^1_{E/\mathfrak{o}_K})$ embeds via the formal expansion, we must have a direct sum decomposition $H \otimes K = (U \otimes K) \oplus \mathrm{H}^0(E_K, \Omega^1_{E_K/K})$.