On the Euler characteristics of complete intersections in an algebraic torus

1 Introduction

Let k be an algebraically closed field. Let $T = \mathbb{G}_{\mathrm{m}}^{n+d}$ be an $(n+d)$-dimensional algebraic torus. Let V be the closed subvariety of T cut out by Laurent polynomials $f_1, \ldots, f_d \in k[x_1^{\pm 1}, \ldots, x_{n+d}^{\pm 1}]$.

Theorem. *In the above situation, if we assume in addition that* $\dim V = n$*, then we have*

$$(-1)^n \chi(V; \mathbb{Q}_\ell) \geq 0.$$

Here, ℓ is any prime number that is different from the characteristic of k, and

$$\chi(V; \mathbb{Q}_\ell) = \sum (-1)^i \dim_{\mathbb{Q}_\ell} \mathrm{H}^i(V; \mathbb{Q}_\ell) = \sum (-1)^i \dim_{\mathbb{Q}_\ell} \mathrm{H}_c^i(V; \mathbb{Q}_\ell)$$

is the ℓ-adic Euler characteristic of V.

When $k = \mathbb{C}$ is the field of complex numbers, this result is due to Loeser and Sabbah; for an arbitrary k, the result is due to Loeser and Gabber. The theorem also follows from the generic vanishing theorem for T. Our purpose is give a p-adic proof of this theorem.

By the standard spread-out argument, we are reduced to the case when k is the algebraic closure of $\mathbb{F}_q$, where q is some power of prime number p, $f_1, \ldots, f_d$ are defined over $\mathbb{F}_q$. In this case, we consider the zeta function of the scheme $V_0 = \operatorname{Spec} \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]/(f_1, \ldots, f_d)$:

$$Z(V_0; t) = \exp \left\{ \sum_{m=1}^{\infty} V_0(\mathbb{F}_{q^m}) \frac{t^m}{m} \right\}.$$

By the classical theorem of Dwork and Grothendieck, $Z(V_0; t)$ is a rational function, so we can write

$$Z(V_0; t)^{(-1)^{n-1}} = \frac{(1-\omega_1 t) \cdots (1-\omega_r t)}{(1-\eta_1 t) \cdots (1-\eta_s t)}. \qquad (*)$$

Moreover, by the cohomological interpretation of the zeta function, $r - s = (-1)^n \chi(V; \mathbb{Q}_\ell)$. So it suffices to prove $r - s \geq 0$

The basis of the argument is the following elementary lemma due to Bombieri.

Lemma (Bombieri)**.** *Let* $\{\theta_1, \ldots, \theta_r\}$ *and* $\{\gamma_1, \ldots, \gamma_s\}$ *be a algebraic numbers. Let* $c(m) \geq 1$, $m = 0, 1, \ldots$ *be an increasing sequence of natural numbers. Consider* p*-adic entire functions*

$$D_1(t) = \prod_{i=1}^{r} \prod_{m=0}^{\infty} (1 - q^m \theta_i t)^{c(m)},$$

$$D_2(t) = \prod_{j=1}^{s} \prod_{m=0}^{\infty} (1 - q^m \gamma_j t)^{c(m)}.$$

If the ratio $D_1(t)/D_2(t)$ *is a* p*-adic entire function, then* $r \geq s$*. Moreover, for each* $j = 1, \ldots, s$*, there is* $m_j \geq 1$ *and* $i_j \in \{1, \ldots, r\}$*, such that*

$$\gamma_j = q^{m_j} \theta_{i_j}.$$

We now construct specific D_1 and D_2 using the zeta function of V_0 expressed as in $(*)$. Letting $c(m) = \binom{n+d+m}{m}$, we define

$$D_1(t) = \prod_{i=1}^{r}\prod_{m=0}^{\infty}(1-q^{d+m}\omega_i)^{c(m)},$$
$$D_2(t) = \prod_{j=1}^{s}\prod_{m=0}^{\infty}(1-q^{d+m}\eta_j)^{c(m)}.$$

Therefore, in view of Bombieri's lemma, it suffices to show that $D_1(t)/D_2(t)$ is a p-adic entire function.

Let me explain the rationale behind the definition of $D_1(t)/D_2(t)$. Let δ be Dwork's décalage operator, defined as

$$h(t)^\delta \xlongequal{\text{def}} \frac{h(t)}{h(qt)},$$

where $h \in 1 + t\overline{\mathbb{Q}}_p[\![t]\!]$ is any formal power series with coefficients in an algebraic closure of $\mathbb{Q}_p$. Then by an easy induction, we find that

$$Z(V_0; q^d t)^{(-1)^{n-1}} = \left\{\frac{D_1(t)}{D_2(t)}\right\}^{\delta^{n+d}}. \qquad (*')$$

We then employ Dwork–Monsky theory to prove the following.

Proposition. *There is a nuclear module (M,α) with characteristic power series $P(M;t)$, such that*

$$Z(V_0; q^d t)^{(-1)^{n-1}} = P(M;t)^{\delta^{n+d}}.$$

Comparing the proposition with $(*')$, using the fact δ is bijective on the set $1 + t\overline{\mathbb{Q}}_p[\![t]\!]$, we conclude immediately that $P(M;t) = D_1(t)/D_2(t)$. By the theory of Dwork and Monsky, we find that $P(M;t)$ is a p-adic entire function. This completes the proof of the theorem. In the remainder of this note, we explain the proof of the Proposition.

2 Dwork theory

Let K be the finite extension of $W(\mathbb{F}_q)[p^{-1}]$ adjoining a nontrivial p^{th} root of unity. Let $\mathcal{O}_K$ be the ring of integers of K. Then $\mathcal{O}_K$ has a uniformizer π satisfying

$$\pi^{p-1} + p = 0.$$

Let

$$B = \mathcal{O}_K[x_1^{\pm 1}, \dots, x_{n+d}^{\pm 1}, y_1, \dots, y_d]^\dagger \otimes_{\mathcal{O}_K} K$$

be the weak completion of the coordinate ring of $\mathbb{G}_{\mathrm{m},\mathcal{O}_K}^{n+d} \times \mathbb{A}_{\mathcal{O}_K}^d$, with π inverted. For each subset I of $\{1,\dots,d\}$, let $B_I = (\prod_{j\in I} y_j)\cdot B$.

Let $F_i \in \mathcal{O}_K[x_1^{\pm 1}, \dots, x_{n+d}^{\pm 1}]$ be the Teichmüller lift of f_i and let

$$G = \sum_{i=1}^{d} y_i F_i \in \mathcal{O}_K[x_1^{\pm 1}, \dots, x_{n+d}^{\pm 1}, y_1, \dots, y_d].$$

We define the following differential operators:

$$D_i = x_i\frac{\partial}{\partial x_i} + \pi x_i \frac{\partial G}{\partial x_i}, \quad i = 1,\dots,n+d$$
$$E_j = y_j\frac{\partial}{\partial y_j} + \pi y_j F_j, \quad j = 1,\dots,d.$$

Using these differential operators, we form a double complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
B^{\oplus\binom{n+d}{2}} & \longrightarrow & \bigoplus_{\mathrm{Card}\, I=1} B_I^{\oplus\binom{n+d}{2}} & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{\mathrm{Card}\, I=m} B_I^{\oplus\binom{n+d}{2}} & \longrightarrow \cdots \longrightarrow & B_{\{1,\ldots,d\}}^{\oplus\binom{n+d}{2}} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
B^{\oplus(n+d)} & \longrightarrow & \bigoplus_{\mathrm{Card}\, I=1} B_I^{\oplus(n+d)} & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{\mathrm{Card}\, I=m} B_I^{\oplus(n+d)} & \longrightarrow \cdots \longrightarrow & B_{\{1,\ldots,d\}}^{\oplus(n+d)} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
B & \longrightarrow & \bigoplus_{\mathrm{Card}\, I=1} B_I & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{\mathrm{Card}\, I=m} B_I & \longrightarrow \cdots \longrightarrow & B_{\{1,\ldots,d\}}
\end{array}$$

where the zeroth row is the Koszul complex of $E_1, \ldots, E_r$, the zeroth column is the Koszul complex of $D_1, \ldots, D_{n+d}$. The i^{th} row is the $\binom{n+d}{i}$-fold direct sum of the zeroth row.

The total complex of this double complex is known as the "overconvergent Dwork complex", and its cohomology is isomorphic to the dual of the rigid cohomology of V_0. Since all we need is an equality of zeta functions, there is no need to quote these fancy theorems, the following Dwork trace formula suffices.

On each space B_I, we have a Dwork operator α_I acting. Thus each entry $E_0^{i,j}$ of the double complex is equipped with a Dwork operator $\alpha^{i,j}$. We have the following Dwork trace formula

$$Z(V_0; q^d t) = \prod_{i,j} \det(1 - tq^{d+n-i-j}\alpha^{i,j})^{(-1)^{i+j-1}}.$$

Letting $M = B_{\{1,\ldots,d\}} / \sum_{j=1}^{d} D_j B_{\{1,\ldots,\widehat{j},\ldots,d\}}$, then M, with the induced action α, is a nuclear module. In [WZ], we explained that the zeroth row, hence every row, is exact except in top degree. Thus, the E_1-page of the spectral sequence associated to the double complex above consists of only one nonzero column, the d^{th} column, whose transpose is

$$M \to M^{\oplus(n+d)} \to M^{\binom{n+d}{2}} \to \cdots \to M,$$

where the last M is placed at the cohomological degree $n + 2d$. It follows that

$$Z(V_0; q^d t)^{(-1)^{n-1}} = \det(1 - t\alpha | M)^{\delta^{n+d}}.$$

This finishes the proof of the proposition.