MEROMORPHIC CONNECTIONS

Contents

In dimension 1, the study of holonomic $\mathcal{D}$-modules is almost reduced to those of meromorphic connections (this remains partly true in higher dimensions, but only partly). A little paradoxically, it is this "almost" restriction which imposes the use of heavy machinery, namely $\mathcal{D}$-modules and derived categories...

— Malgrange

References.

- A. Grothendieck. *On the de Rham cohomology of algebraic varieties*. Publications mathématiques de l'I.H.É.S., tome 29 (1966), p. 95-103.
- F. J. Castro-Jiménez. *Exercices sur le complexe de De Rham et l'image directe des $\mathcal{D}$-modules*. In: Maisonobe and Sabbah (Ed.), Éléments de la théorie des systèmes différentiels. Images directes et constructibilité (1993).

1. Meromorphic localization

In algebraic geometry, restricting a coherent sheaf to a distinguished open is a matter of inverting a function. This is not the case in complex analytic geometry: the set of holomoprhic functions on $\mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ is much larger than $\mathcal{O}(\mathbb{A}^{1,\mathrm{an}})[t^{-1}]$. From an algebraic point of view, the latter set is more manageable than the former.

Fix the following notation.

- Let X be a complex analytic manifold.
- Let Y be a closed analytic subspace of X, $\mathrm{U} = \mathrm{X} \setminus \mathrm{Y}$.
- Let $j\colon \mathrm{U} \to \mathrm{X}$ be the open embedding, and let $i\colon \mathrm{Y} \to \mathrm{X}$ be the closed embedding.

We shall define the "meromorphic localization" of an $\mathcal{O}_{\mathrm{X}}$-module on X to U.

1.1. Localization. For a $\mathcal{O}_{\mathrm{X}}$-module M on X, its *localization* on U is defined as

$$\mathrm{M}[*\mathrm{Y}] = \operatorname*{colim}_{k\to\infty} \mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}_{\mathrm{Y}}^k, \mathrm{M}).$$

Its total derived functor is denoted by

$$\mathbb{R}\mathrm{M}[*\mathrm{Y}] = \operatorname*{colim}_{k\to\infty} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}_{\mathrm{Y}}^k, \mathrm{M})$$

(Kashiwara (1978) denoted it by $\mathbb{R}\Gamma_{[\mathrm{X}|\mathrm{Y}]}(\mathrm{M})$, and its cohomology sheaves by $\mathcal{H}^{\bullet}_{[\mathrm{X}|\mathrm{Y}]}\mathrm{M}$).

Created on January 19, 2022. Compiled on January 20, 2022.

By Nullstellensatz, $\mathrm{M}[*\mathrm{Y}]$ and $\mathbb{R}[*\mathrm{Y}]$ only depend on the reduced analytic space $\mathrm{Y}_{\mathrm{red}}$, not on the ideal $\mathcal{I}_{\mathrm{Y}}$.

The algebraic geometry analogue of $\mathbb{R}\mathrm{M}[*\mathrm{Y}]$ is $\mathcal{F} \mapsto \mathbb{R}j_*j^*\mathcal{F}$, where $j\colon \mathrm{U} \to \mathrm{X}$ is an open embedding of schemes, $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathrm{X}}$-module. When j is an affine morphism, $\mathrm{R}j_* = j_*$; but in general one cannot expect the higher direct images to vanish. See Example 1.2, whose argument works both in the analytic and algebraic context.

When Y is a Cartier divisor, the powers of $\mathcal{I}_{\mathrm{Y}} = \mathcal{O}_{\mathrm{X}}(-\mathrm{Y})$ are invertible sheaves, whence $\mathcal{E}xt^j_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}^k_{\mathrm{Y}}, -) = 0$ for $j > 0$. Thus $\mathrm{M}[*\mathrm{Y}]$ is simply $\bigcup_{k=0}^{\infty} \mathrm{M} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{O}_{\mathrm{X}}(k\mathrm{Y})$, and

$$\mathbb{R}\mathrm{M}[*\mathrm{Y}] = \operatorname*{colim}_{k\to\infty} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}^k_{\mathrm{Y}}, \mathrm{M}) = \mathrm{M}[*\mathrm{Y}], \quad \text{when Y is a Cartier divisor.}$$

When Y is a Cartier divisor, $\mathcal{O}_{\mathrm{X}}[*\mathrm{Y}]$ is also a sheaf of coherent rings; as locally it is isomorphic to $\mathcal{O}_{\mathrm{X}}[\mathrm{T}]/(\mathrm{T}f - 1)$.

1.2. Example. If Y has codimension two or higher, the higher derived functor are not zero. In the situation above assume $\mathrm{X} = \mathbb{C}^2$, $\mathrm{Y} = \{0\}$. Then $\mathcal{I}_{\mathrm{Y}} = (x, y) \cdot \mathcal{O}_{\mathrm{X}}$. Instead of using $\mathcal{I}^k_{\mathrm{Y}}$ to calculate the localization, we use the ideals $\mathcal{J}_k = (x^k, y^k)$ instead, as the systems

$$[\cdots \subset \mathcal{J}_{k+1} \subset \mathcal{J}_k \subset \cdots] \quad \text{and} \quad \left[\cdots \subset \mathcal{I}^{k+1}_{\mathrm{Y}} \subset \mathcal{I}^k_{\mathrm{Y}} \subset \cdots\right]$$

are cofinal to each other. Hence, the localization $\mathbb{R}\mathcal{O}_{\mathrm{X}}[*\mathrm{Y}]$ could be computed by

$$\operatorname*{colim}_{k\to\infty} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(\mathcal{J}_k, \mathcal{O}_{\mathrm{X}}).$$

The system $(\mathcal{J}_k)_{k=1}^{\infty}$ admits a free resolution

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathrm{X}} & \xrightarrow{[-y^2, x^2]^{\mathsf{T}}} & \mathcal{O}^2_{\mathrm{X}} & \longrightarrow & \mathcal{J}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle xy} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathrm{X}} & \xrightarrow{[-y, x]^{\mathsf{T}}} & \mathcal{O}^2_{\mathrm{X}} & \longrightarrow & \mathcal{J}_1 & \longrightarrow & 0
\end{array}\ .$$

Taking $\mathcal{H}om_{\mathcal{O}_{\mathrm{X}}}(-, \mathcal{O}_{\mathrm{X}})$ and taking colimits, using the fact that

$$\operatorname*{colim}_{k\to\infty} \left[\mathrm{A} \xrightarrow{a} \mathrm{A} \xrightarrow{a} \mathrm{A} \to \cdots\right] = \mathrm{A}[a^{-1}] \quad (\text{A commutative ring, } a \in \mathrm{A}),$$

we see $\mathbb{R}\mathcal{O}_{\mathrm{X}}[*\mathrm{Y}]$ is computed by the two-term complex

$$(f, g) \mapsto f + g\colon \underset{\bullet}{\mathcal{O}_{\mathrm{X}}[x^{-1}] \oplus \mathcal{O}_{\mathrm{X}}[y^{-1}]} \to \mathcal{O}_{\mathrm{X}}[(xy)^{-1}]$$

The R^0 term is simply $\mathcal{O}_{\mathrm{X}}$, which surely follows from the Hartogs theorem, the R^1 term is visibly supported at the origin.

1.3. Local cohomology. The *meromorphic local cohomology* of M supported on Y is defined to be

$$\mathbb{R}\Gamma_{[\mathrm{Y}]}(\mathrm{M}) = \operatorname*{colim}_{k\to\infty} \mathbb{R}\mathcal{H}om_{\mathcal{O}_\mathrm{X}}(\mathcal{O}_\mathrm{X}/\mathcal{I}_\mathrm{Y}^k, \mathrm{M}).$$

Taking $\mathbb{R}\mathcal{H}om_{\mathcal{O}_\mathrm{X}}(-,\mathrm{M})$ on the exact sequence

$$0 \to \mathcal{I}_\mathrm{Y}^k \to \mathcal{O}_\mathrm{X} \to \mathcal{O}_\mathrm{X}/\mathcal{I}_\mathrm{Y}^k$$

gives rise to a distinguished triangle of $\mathcal{O}_\mathrm{X}$-modules

$$\mathbb{R}\Gamma_{[\mathrm{Y}]}\mathrm{M} \to \mathrm{M} \to \mathbb{R}\mathrm{M}[*\mathrm{Y}] \to .$$

The algebraic geometry analogue of this is the "standard triangle" $i_*i^!\mathcal{F} \to \mathcal{F} \to \mathbb{R}j_*j^*\mathcal{F}$.

Of course, we can equally apply the standard triangle to the $\mathcal{O}_\mathrm{X}$-module M:

$$i_*i^!\mathrm{M} \to \mathrm{M} \to \mathbb{R}j_*j^*\mathrm{M} \to .$$

The complex $i_*i^!\mathrm{M}$, also denoted by $\mathbb{R}\Gamma_\mathrm{Y}\mathrm{M}$, is the total derived functor of the functor $\mathrm{M} \mapsto \Gamma_\mathrm{Y}(\mathrm{M})$, where $\Gamma_\mathrm{Y}(\mathrm{M})$ is the space of sections of M which are supported on Y. This "brutal" triangle and the "meromorphic" triangle are compatible, in that the following diagram is commutative:

$$\begin{array}{ccccccc}
\mathbb{R}\Gamma_{[\mathrm{Y}]}\mathrm{M} & \longrightarrow & \mathrm{M} & \longrightarrow & \mathbb{R}\mathrm{M}[*\mathrm{Y}] & \longrightarrow & \\
\downarrow & & \| & & \downarrow & & . \\
\mathbb{R}\Gamma_\mathrm{Y}\mathrm{M} & \longrightarrow & \mathrm{M} & \longrightarrow & \mathbb{R}j_*j^{-1}\mathrm{M} & \longrightarrow &
\end{array}$$

2. Meromorphic connections

In this section, D is a *divisor* on a complex manifold X. Let $j\colon \mathrm{U} = \mathrm{X}\setminus\mathrm{D} \to \mathrm{X}$ be the open immersion of the complement of D, and let $i\colon \mathrm{D} \to \mathrm{X}$ be the closed immersion.

The reason that we restrict to the divisor case is due to the lack of proper language, namely the language of $\mathcal{D}$-modules.

2.1. Definition of a meromorphic connection. An *integrable connection on* U*, meromorphic along* D, is a locally free $\mathcal{O}_\mathrm{X}[*\mathrm{D}]$-module M on X, together with a $\mathbb{C}$-linear map

$$\nabla\colon \mathrm{M} \to \Omega_\mathrm{X}^1 \otimes_{\mathcal{O}_\mathrm{X}} \mathrm{M}$$

subject to the usual Leibniz rule, and the integrable condition $\nabla^2 = \{0\}$.

When without ambiguity, I shall simply say "M is a meromorphic connection on X" and tacitly assume the integrablity.

To understand this notion, we work locally, and choose an $\mathcal{O}_\mathrm{X}[*\mathrm{D}]$-basis $(e_1,\dots,e_r)$ of M. Then we can write

$$\nabla(e_1,\dots,e_r) = (e_1,\dots,e_r)\cdot \mathrm{A}$$

where A is a matrix of meromorphic 1-forms on X with poles along D only. The matrix A being holomorphic on U, the meromorphic connection M could also be regarded as an integrable connection on U — this *is* the inverse image $j^{-1}\mathrm{M}$.

Let M and N be two meromorphic connections on X, both meromorphic along the divisor D. A morphism of $\mathcal{O}_\mathrm{X}[*\mathrm{D}]$-modules $\varphi\colon \mathrm{M} \to \mathrm{N}$ is said to be *horizontal*, or *compatible* with the connections, if $j^{-1}\varphi$ is horizontal. We say M and N are *isomorphic as meromorphic connections*, if there exists a horizontal $\mathcal{O}_\mathrm{X}[*\mathrm{D}]$-module isomorphism between them.

Although M and j^{-1}M are intimately related, the structure of M is more refined than j^{-1}M. In particular, j^{-1}M and j^{-1}N are isomorphic as integrable connections on U does not imply M and N are isomorphic. Consider for example $\mathrm{X} = \mathbb{C}$, $\mathrm{D} = \{0\}$, and

$$\nabla\colon \mathcal{O}_{\mathrm{X}}[*\mathrm{D}] \to \mathcal{O}_{\mathrm{X}}[*\mathrm{D}]\mathrm{d}t, \quad \nabla(f) = \mathrm{d}f + t^{-2}f(t)\mathrm{d}t.$$

The above formula defines an integrable connection M on $\mathbb{C} \setminus \{0\}$, meromorphic along $\{0\}$. Note that j^{-1}M is isomorphic to the trivial connection $(\mathcal{O}_{\mathrm{U}}, \mathrm{d})$, which is $j^{-1}(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}], \mathrm{d})$. But M and $(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}], \mathrm{d})$ are not isomorphic, as $\mathrm{Ker}(\nabla_{\mathrm{d}/\mathrm{d}t}\colon \mathrm{M} \to \mathrm{M})$ is trivial, whereas $\mathrm{Ker}(\mathrm{d}\colon \mathcal{O}_{\mathrm{X}}[*\mathrm{D}] \to \mathcal{O}_{\mathrm{X}}[*\mathrm{D}]\mathrm{d}t)$ is isomorphic to $\mathbb{C}$.

2.2. The de Rham complex. Let n be the dimension of X. Let M be a meromorphic connection on X. Then the covariant derivative ∇ extends to a map $\nabla\colon \Omega^p_{\mathrm{X}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M} \to \Omega^{p+1}_{\mathrm{X}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M}$ by "forcing the Leibniz rule". The complex

$$DR(\mathrm{M}): \quad 0 \to \underset{\bullet}{\mathrm{M}} \xrightarrow{\nabla} \Omega^1_{\mathrm{X}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_{\mathrm{X}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{M} \to 0$$

is called the *de Rham complex* of M. The dot underneath M means that we put M in cohomological degree 0 in this complex.

The *de Rham cohomology* of M is then defined to be the hypercohomology of the de Rham complex: $\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}, \mathrm{M}) = \mathbb{R}\Gamma^i(\mathrm{X}, DR(\mathrm{M}))$.

2.3. De Rham complex without meromorphic structure. The simplest situation is when $\mathrm{D} = \varnothing$, i.e., no meromorphic structure is concerned. Thus the notion of a meromorphic connection is the same as a plain integrable connection on X. Under these hypotheses we have the following result.

Theorem 1. *Let* X *be a complex manifold. Let* M *be an integrable connection on* X*. Let* E *be the local system associated to* M*. Then the natural morphism of chain complexes*

$$\iota\colon \mathrm{E}[0] \to DR(\mathrm{M})$$

is a quasi-isomorphism. In particular, there is a canonical isomorphism $\mathrm{H}^i(\mathrm{X}, \mathrm{E}) \simeq \mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}, \mathrm{M})$.

Proof. The "trivial" case is when $\mathrm{M} = (\mathcal{O}_{\mathrm{X}}, \mathrm{d})$ is the trivial integrable connection. Then $\mathrm{E} = \mathbb{C}_{\mathrm{X}}$ and $DR(\mathcal{O}_{\mathrm{X}})$ is the usual de Rham complex of holomorphic forms on X. The theorem then reduces to the *Poincaré lemma*, which I reproduce in Theorem 2 below.

The general case follows from the "trivial" case. This is because the map ι is defined *globally*; hence checking it is a quasi-isomorphism is a *local* problem. By restricting to polydisks and applying Cauchy's theorem (i.e., M is locally isomorphic to $(\mathcal{O}^r, \mathrm{d})$), we are reduced to the trivial case. □

Theorem 2 (Poincaré lemma)**.** *Let* Δ^n *be an n-dimensional poldisk. Then the natural chain map* $\mathbb{C}[0] \to \Gamma(\Delta^n, DR(\mathcal{O}))$ *is a quasi-isomorphism.*

Proof. Since Δ^n is Stein, I will not be very careful distinguishing coherent sheaves and their sections over Δ^n.

Let $\Omega^i_{>p}$ be the set of i-forms on Δ^n which only involves $\mathrm{d}z_p, \ldots, \mathrm{d}z_n$. Define $d_p : \Omega^i_{>p} \to \Omega^{i+1}_{>p}$ to be the differential which only differentiate $z_{p+1}, \ldots, z_n$. Thus d_1 equals the usual exterior differential.

Define a filtration $\mathrm{L}^\bullet$ on $DR(\mathcal{O})$ as follows

$$
\begin{array}{lcccccccccccc}
\mathrm{L}^0: & \mathcal{O} & \xrightarrow{d_0} & \Omega^1_{>0} & \xrightarrow{d_0} & \Omega^2_{>0} & \longrightarrow & \cdots & \longrightarrow & \Omega^{n-1}_{>0} & \xrightarrow{d_0} & \Omega^n_{>0} & \\
& & & {\scriptstyle \mathrm{d}z_1\wedge}\uparrow & & \uparrow{\scriptstyle \wedge \mathrm{d}z_1} & & & & {\scriptstyle \wedge \mathrm{d}z_1}\uparrow & & \uparrow{\scriptstyle \wedge \mathrm{d}z_1} & \\
\mathrm{L}^1: & & & \mathcal{O} & \xrightarrow{d_1} & \Omega^1_{>1} & \longrightarrow & \cdots & \longrightarrow & \Omega^{n-2}_{>1} & \xrightarrow{d_1} & \Omega^{n-1}_{>1} & \\
& & & & & & & & & \uparrow & & \uparrow & \\
\vdots & & & & & & & \ddots & & \vdots & & \vdots & . \\
& & & & & & & & & {\scriptstyle \wedge \mathrm{d}z_{n-1}}\uparrow & & \uparrow{\scriptstyle \wedge \mathrm{d}z_{n-1}} & \\
\mathrm{L}^{n-1} & & & & & & & & & \mathcal{O} & \xrightarrow{d_{n-1}} & \Omega^1_{>n-1} & \\
& & & & & & & & & & & \uparrow{\scriptstyle \wedge \mathrm{d}z_n} & \\
\mathrm{L}^n & & & & & & & & & & & \mathcal{O} &
\end{array}
$$

Let $\mathcal{O}_i = \mathrm{pr}^{-1}_{[1,i]}\mathcal{O}_{\Delta^i}$, where $\mathrm{pr}_{[1,i]} \colon \Delta^n \to \Delta^i$ is the projection $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_i)$. We prove by reverse induction that the natural inclusion $\mathcal{O}_i[-i] \to \mathrm{L}^i$ is a quasi-isomorphism. Note the assertion for $i = 0$ implies the theorem.

The $i = n$ case is trivial. In general, by definition, we have

$$\mathrm{Gr}^i_{\mathrm{L}} DR(\mathcal{O}) \simeq \mathrm{L}^{i+1}[1].$$

Therefore by taking cohomology sheaf to the short exact sequence $0 \to \mathrm{L}^{i+1} \to \mathrm{L}^i \to \mathrm{Gr}^i_{\mathrm{L}}$ and applying the inductive hypothesis gives $\mathrm{H}^j(\mathrm{L}^i) = 0$ for $j > i + 1$, and an exact sequence

$$0 \to \mathrm{H}^i(\mathrm{L}^i) \to \mathcal{O}_{i+1} \xrightarrow{\delta} \mathcal{O}_{i+1} \to \mathrm{H}^{i+1}(\mathrm{L}^i) \to 0.$$

Recall the definition of the boundary map δ: first we lift a function f in $\mathcal{O}_{i+1}$ to a function in $\mathcal{O}$, which can be taken f itself, which is independent of the variables $z_{i+2}, \ldots, z_n$; then apply d_i to f, which will only differentiates z_{i+1} as the lifted f is independent of $z_{i+2}, \ldots, z_n$; finally, $\delta(f) = d_i(f)/\mathrm{d}z_{i+1} = \partial f/\partial z_{i+1}$. In other words, δ is given by $\partial/\partial z_{i+1}$. This map is obviously surjective, and its kernel is $\mathcal{O}_i$. This finishes the induction and completes the proof. □

2.4. De Rham cohomology of trivial meromorphic connections (nonsingular case). In general, the meromorphic de Rham cohomology of a meromorphic connection can be subtle. Later we will see that there is a dichotomy: when M is regular along D (warning: this means the connection of M has regular singularity along D; it does not mean ∇ extends to a holomorphic connection on X), then the de Rham cohomology of M is the same as the de Rham cohomology of $j^{-1}\mathrm{M}$; when M is irregular the de Rham cohomology of M does not agree with that of $j^{-1}\mathrm{M}$ in general.

For now, let us examine the simplest regular meromorphic connection, namely the trivial one. Even in this trivial case we are not quite ready to give a full answer (due to the limitation of language, if not the difficulty of the problem), and we have to assume D is nonsingular.

Theorem. *Let* D *be a nonsingular divisor of a complex manifold* X*. Then the natural map*

$$DR(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}]) \to \mathbb{R}j_* DR(\mathcal{O}_{\mathrm{U}})$$

is a quasi-isomorphism. In particular,

$$\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}, \mathcal{O}_{\mathrm{X}}[*\mathrm{D}]) \simeq \mathrm{H}^i_{\mathrm{DR}}(\mathrm{U}).$$

Proof. Since $j^{-1}\mathcal{O}_{\mathrm{X}}[*\mathrm{D}] = \mathcal{O}_{\mathrm{U}}$, we have $j^{-1}DR(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}]) \simeq DR(\mathcal{O}_{\mathrm{U}})$. The "natural map" mentioned in the statement is the one induced from this by adjunction.

Since $\mathrm{U} \to \mathrm{X}$ is an a Stein morphism, $\mathbb{R}^i j_*\mathcal{F} = 0$ for any coherent sheaf on U. In particular, $\mathbb{R}j_*DR(\mathcal{O}_{\mathrm{U}}) = \mathbb{R}^0 j_*DR(\mathcal{O}_{\mathrm{U}})$.

The theorem is a local one, it suffices to assume $\mathrm{X} = \Delta^n$ is a polydisk, and D is the hyperplane defined by $z_1 = 0$, and it suffices to prove the natural inclusion

$$\Gamma(\mathrm{X}, DR(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}])) \hookrightarrow \Gamma(\mathrm{U}, DR(\mathcal{O}_{\mathrm{U}}))$$

is a quasi-isomorphism.

Let us consider the following commutative diagram

$$(*) \qquad \begin{array}{ccc} \left[\mathbb{C} \xrightarrow{0} \mathbb{C}\frac{\mathrm{d}z_1}{z_1}\right] & \hookrightarrow & \Gamma(\mathrm{U}, DR(\mathcal{O}_{\mathrm{U}})) \\ \big\downarrow & \nearrow & \\ \Gamma(\mathrm{X}, DR(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}])) & & \end{array} .$$

As U is isomorphic to $\Delta^* \times \Delta^{n-1}$, which is homotopy equivalent to $\mathbb{S}^1$, elementary topology implies the first horizontal arrow is a quasi-isomorphism. It thus suffices to prove the vertical arrow is a quasi-isomorphism.

Write $\mathrm{X} = \Delta \times \mathrm{S}$. Let K be the following complex

$$\mathcal{O}_{\mathrm{X}}[z_1^{-1}] \xrightarrow{d_s} \mathrm{pr}_2^*\Omega^1_{\mathrm{S}}[z_1^{-1}] \xrightarrow{d_s} \cdots \to \mathrm{pr}_2^*\Omega^n_{\mathrm{S}}[z_1^{-1}],$$

where d_s are differentials which only takes derivatives with respect to $z_2, \ldots, z_n$. As a complex it is an infinite direct sum

$$\bigoplus_{k=0}^{\infty} z_1^{-k}\left[\mathcal{O}_{\mathrm{X}} \xrightarrow{d_s} \mathrm{pr}_2^*\Omega_1 \to \cdots\right]$$

of the relative de Rham complex for the relative connection on $\mathcal{O}_{\mathrm{X}}$. By a relative version of the Poincaré lemma, we know this complex is quasi-isomorphic to the plain vector space $\mathcal{O}(\Delta)[z_1^{-1}]$.

Now we note that $\Gamma(\mathrm{X}, DR(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}]))$ fits into an exact sequence of complexes

$$0 \to \mathrm{K}[1] \xrightarrow{\mathrm{d}z_1\wedge} \Gamma(\mathrm{X}, DR(\mathcal{O}_{\mathrm{X}}[*\mathrm{D}])) \to \mathrm{K} \to 0,$$

the second map being simply killing differentials with $\mathrm{d}z_1$ appearing. Taking cohomology yields a long exact sequence:

$$0 \to \mathrm{H}^0_{\mathrm{DR}} \to \mathcal{O}(\Delta)[z_1^{-1}] \xrightarrow{\partial/\partial z_1} \mathcal{O}(\Delta)[z_1^{-1}]\mathrm{d}z_1 \to \mathrm{H}^1_{\mathrm{DR}} \to 0,$$

and we deduce the vanishing of de Rham cohomology of $\mathcal{O}_{\mathrm{X}}[*\mathrm{D}]$ in degree ≥ 2.

By a diagram chasing, one checks that the map in the middle is indeed $\partial/\partial z_1$. This check is identical to the check made in proof of Poincaré lemma (§2.3, Theorem 2). Finally, one performs a direct computation to show

$$\mathrm{H}^0_{\mathrm{DR}} \cong \mathbb{C}, \quad \text{and} \quad \mathrm{H}^1_{\mathrm{DR}} \cong \mathbb{C}\frac{\mathrm{d}z}{z}.$$

This shows that the vertical arrow in $(*)$ does indeed induce an isomorphism on the level of cohomology. This finishes the proof. □

2.5. Algebraic de Rham cohomology. Let U be a smooth algebraic variety over a field k of characteristic 0. Then one can form the *algebraic de Rham complex* of U

$$DR(\mathcal{O}_{\mathrm{U}}) : \mathcal{O}_{\mathrm{U}} \to \Omega^1_{\mathrm{U}} \to \cdots \to \Omega^n_{\mathrm{U}},$$

and define the algebraic de Rham cohomology to be the hypercohomology of $DR(\mathcal{O}_{\mathrm{U}})$:

$$\mathrm{H}^i_{\mathrm{DR}}(\mathrm{U}/k) = \mathbb{R}^i\Gamma(\mathrm{U}, DR(\mathcal{O}_{\mathrm{U}})).$$

Grothendieck has shown the following

Theorem. *Let* $\iota\colon k \to \mathbb{C}$ *be an embedding. Then there is a natural isomorphism*

$$\mathrm{H}^\bullet_{\mathrm{DR}}(\mathrm{U}/k) \otimes_k \mathbb{C} \simeq \mathrm{H}^\bullet(\mathrm{U}^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

A direct verification of the theorem can be made when $\mathrm{U} = \mathbb{A}^1 \setminus \{0\}$. In this case, the de Rham complex is simply

$$\bigoplus_{m\in\mathbb{Z}} k \cdot t^m \xrightarrow{\mathrm{d}} \bigoplus_{m\in\mathbb{Z}} k \cdot t^m \frac{\mathrm{d}t}{t}$$

which is an infinite direct sum of the complexes $k \xrightarrow{m} k$ when m runs in $\mathbb{Z}$. The complex is thus quasi-isomorphic to

$$k \xrightarrow{0} k \cdot \frac{\mathrm{d}t}{t}$$

which is the correct answer.

We will prove the theorem (and its generalization to twisted coefficients, provided the coefficient is a *regular*) later. For now, let us assume that there is an open immersion $j\colon \mathrm{U} \to \mathrm{X}$ into a nonsingular projective variety X, such that its complement is a *nonsingular* divisor D. By base change, we might as well assume $k = \mathbb{C}$.

Note that $\{j_*\mathrm{DR}(\mathcal{O}_{\mathrm{U}})\}^{\mathrm{an}}$ equals $DR(\mathcal{O}_{\mathrm{X}^{\mathrm{an}}}[*\mathrm{D}^{\mathrm{an}}])$, and its entries are all filtered colimits of coherent sheaves on X^{an}. Therefore by GAGA and Theorem 2.4 we obtain

$$\mathrm{H}^\bullet_{\mathrm{DR}}(\mathrm{U}/\mathbb{C}) \simeq \mathrm{H}^\bullet_{\mathrm{DR}}(\mathrm{X}^{\mathrm{an}}, \mathcal{O}_{\mathrm{X}^{\mathrm{an}}}[*\mathrm{D}^{\mathrm{an}}]) \simeq \mathrm{H}^\bullet(\mathrm{U}, \mathbb{C}).$$

The proof of Grothendieck's theorem follows the same line, provided we can prove Theorem 2.4 for a singular divisor. Granting the singular version of Theorem 2.4, we can argue as follows:

- the theorem being local, we can assume U is affine;
- for affine U we embed it into a projective variety X as a Zariski open;
- applying resolution of singularities, we may assume X is nonsingular (and could even assume $\mathrm{X} \setminus \mathrm{U}$ is a divisor with strictly normal crossing);
- repeat the GAGA argument.