LE THÉORÈME DE MONODROMIE (D'APRÈS BRIESKORN)

Foreword. — This post is a translation of [Del70, III.2], where Deligne exposed Brieskorn's proof of the monodromy theorem. See Theorem 3 below. One feature of this version is that it does not require the properness of the morphism, but only assumes the cohomology of the fibers are locally constant.

The proof is based on the regularity of the Gauß-Manin connection for a 1-parameter family and a theorem of Gel'fond in transcendental number theory. Note that the formulation below does *not* require the family to be proper (thus the local systems do not seem to underly polarized variation of Hodge structure), whereas Borel's proof (exposed in e.g., [Sch73, (4.5)], and this post), being essentially the application of the distance decreasing property of the period mapping, does not seem to apply in the generality presented below. On the other hand, Borel's proof does not need the regularity whatsoever.

One place where Brieskorn's argument applies but Borel's does not is when we consider the vanishing cohomology bundle of the Milnor's fibration of an isolated hypersurface singularity.

The proof can be summarized as follows:

- The regularity of a connection allows us to obtain the eigenvalues of the monodromy operator by means of the residue matrix of the connection.
- The eigenvalues α of the residue of the connection change by a conjugation when we conjugate an embedding of the field of definition into $\mathbb{C}$, but they at any rate are the exponents of the eigenvalues of the monodromy operation.
- The monodromy operation of each conjugated family always preserves an integral lattice coming from Betti cohomology, hence has algebraic eigenvalues.
- So $\exp(2\pi i\sigma(\alpha))$ are always algebraic numbers for any σ whatsoever. But then a cardinality count shows α itself must also be algebraic.
- One then applies Gel'fond's theorem assertion α and $\exp(2\pi i\alpha)$ cannot both be algebraic unless α is rational, and concludes that α is rational, which implies the monodromy is quasi-unipotent.

1. Let S be a smooth algebraic curve, obtained from a smooth projective curve $\overline{S}$ by removing a finite set T of points. For $t \in T$, the *local monodromy group* of t, or the *local fundamental group* of S at t, is the fundamental group of $D - \{t\}$, where D is a small disk centered at t. The group is canonically isomorphic to $\mathbb{Z}$, and we call the canonical generator the "monodromy transformation".

If V is a local system of $\mathbb{C}$-vector spaces on S, the local monodromy group of t acts on $V|_{D-\{t\}}$. If V is the complexifed local system of a finite type $\mathbb{Z}$-module, then the characteristic polynomial of the monodromy transformation is of integral coefficients.

Recall that a linear transformation is said to be *quasi-unipotent* if some of its power is unipotent. A local system of $\mathbb{C}$-vector spaces on S is said to be *quasi-unipotent (resp. unipotent) at infinity* if for all $t \in T$, the corresponding monodromy transformation is quasi-unipotent (resp. unipotent).

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Example 2. Let $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ be the Poincaré upper half plane and Γ a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ such that $\Gamma\backslash X$ has finite volume. We then know that $\Gamma\backslash X$ is an algebraic curve with fundamental group Γ. Each finite dimensional complex representation ρ of Γ defines a local system V_ρ on $\Gamma\backslash X$. To ensure V_ρ to be unipotent at infinity, the following condition is necessary and sufficient: for any $\gamma \in \Gamma$ that is unipotent in $\mathrm{SL}_2(\mathbb{R})$, $\rho(\gamma)$ is unipotent.

Theorem 3. *Let S be as in* 1. *Let i be an integer. Let $f : X \to S$ be a smooth morphism. Suppose that $R^i f_*\mathbb{C}$ is a local system* (*i.e., is locally constant*). *Then $R^i f_*\mathbb{C}$ is quasi-unipotent at infinity.*

The proof is based on the following theorem of Gel'fond [Gel34]:

Theorem 4. *If α and* $\exp(2\pi\mathrm{i}\alpha)$ *are all algebraic numbers, then α is rational.*

An immediate corollary of Theorem 4 is the following.

Corollary 5. *If N is a matrix with coefficients in a subfield K of $\mathbb{C}$, and if for any embedding $\sigma : K \to \mathbb{C}$, the characteristic polynomial of* $\exp(2\pi\mathrm{i}\sigma(N))$ *is of integral coefficients, then* $\exp(2\pi\mathrm{i}N)$ *is quasi-unipotent.*

Proof. Suppose α is an eigenvalue of N in an extension K' of K. For any embedding σ of K' in $\mathbb{C}$, $\sigma(\alpha)$ is an eigenvalue of $\sigma(N)$, and $\exp(2\pi\mathrm{i}\sigma(\alpha))$ is an eigenvalue of $\exp(2\pi\mathrm{i}\sigma(N))$. Since the characteristic polynomial of $\exp(2\pi\mathrm{i}\sigma(N))$ is integral, it follows that $\exp(2\pi\mathrm{i}\sigma(\alpha))$ is an algebraic number.

Next, we claim that α is algebraic. Granted this, Theorem 4 then implies α is rational. It follows that the eigenvalues $\exp(2\pi\mathrm{i}\alpha)$ of $\exp(2\pi\mathrm{i}N)$ are roots of unity, and we win.

If α were transcendental, then for any transcendental number of the form $\mathrm{i}x$, $x \in \mathbb{R}$, there exists σ such that $\sigma(\alpha) = \mathrm{i}x$, so $\exp(2\pi\mathrm{i}\alpha) = \mathrm{e}^{-x}$. On the one hand, since $\exp(2\pi\mathrm{i}\sigma(\alpha))$ is an algebraic number, the cadinality of the set $E = \{\exp(2\pi\mathrm{i}\sigma(\alpha))\}_{\sigma:K'\to\mathbb{C}}$ is countable. On the other hand, $x \mapsto \exp(-x)$ is a bijective map between $\mathbb{R}$ and $\mathbb{R}_{>0}$; since there are infinitely many transcendental x, the totality of

$$S = \{\exp(-x) : x \in \mathbb{R} \text{ is transcendental}\}$$

is uncountable. As we have said before, if α were to be transcendental, then S would be a subset of $E = \{\exp(2\pi\mathrm{i}\sigma(\alpha))\}_{\sigma:K'\to\mathbb{C}}$, implying E is uncountable. This is a contradiction. □

Proof of Theorem 3. By shrinking S, we can assume that $\mathcal{V} = \mathbb{R}^i f_*\Omega^\bullet_{X/S}$ is locally free. If K is a subfield of $\mathbb{C}$, such that $f, X, S, \overline{S}$ and the points in T are defined over K, i.e., come by extension of scalar of $\sigma_0 : K \to \mathbb{C}$, of $f_0 : X_0 \to S_0$, and $T_0 \subset \overline{S}_0(K)$. We know that the Gauß-Manin connection on $\mathcal{V}_0 = \mathbb{R}^i f_{0*}\Omega^\bullet_{X_0/S_0}$ is regular, that there exits a vector bundle $\mathcal{V}_0'$ on $\overline{S_0}$ extending $\mathcal{V}_0$, and that $\mathcal{V}_0'$ admits a connection that has simple poles at $t \in T_0$ (see [Kat70], [Del70, Théorème 7.9], note in the second reference no properness of the morphism is required). Suppose N_t is the residue matrix of the connection at $t \in T_0$, with respect to a basis of $\mathcal{V}'_{0,t}$.

For every embedding $\sigma : K \to \mathbb{C}$, f_0 defines, by extending the scalars, a morphism

$$f_{(\sigma)} : X_{(\sigma)} \to S_{(\sigma)}$$

and $\mathcal{V}_{(\sigma)} = \mathbb{R}^i f_{(\sigma)*}\Omega^\bullet_{X_{(\sigma)}/S_{(\sigma)}}$ is deduced by extending scalars of $\mathcal{V}_0$. According to the "formal Fuchsian theory", $\exp(2\pi \mathrm{i}\sigma(N_t))$ has the same characteristic polynomial as the local monodromy at t against $R^i f_{(\sigma)*}\mathbb{C}$. According to 1, $\exp(2\pi \mathrm{i}\sigma(N_t))$ has a characteristic polynomial with integral coefficients, and by Corollary 5, $\exp(2\pi \mathrm{i} N_t)$ is quasi-unipotent. We win. □

References

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