THE MONSKY–WASHNITZER TOPOS

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Monsky and Washnitzer defined a good "p-adic" cohomology theory for smooth affine varieties [11] over a field of characteristic $p > 0$. For arbitrary k-varieties, Berthelot defined another cohomology theory known as the rigid cohomology, which agrees with the cohomology of Monsky–Washnitzer for smooth affine k-varieties. The original definition of rigid cohomology was not site-theoretic. For a proper k-variety X, Ogus defined a "convergent site" [12], and the cohomology of a certain sheaf on this site agrees with the rigid cohomology of X. For arbitrary X, Le Stum had defined an "overconvergent site" [9], and cohomology of a sheaf on this site computes the rigid cohomology of X. In this paper, we define, for any k-variety X, another site, which we call the *Monsky–Washnitzer site* of X, and we prove the cohomology of a sheaf on this site also computes the rigid cohomology of X. Comparing with Le Stum's overconvergent site, we feel the Monsky–Washnitzer site is more related to the construction of Monsky–Washnitzer (whereas Le Stum's site is more related to Berthelot's original theory).

The idea is to mimic the every construction made by Ogus in his convergent theory [12] (there is also Shiho's logarithmic version of Ogus's theory [14]), with formal schemes replaced by weak formal schemes of Meredith [10]. According to Gronthendieck's understanding [8, §2.3], it seems such a theory has been perceived by Monsky and Washnitzer. Although Grothendieck commented that the method of Monsky–Washnitzer is "too closely bound to differential forms, which practically limits its applications to smooth schemes", the difficulty is overcome by embedding the scheme in question into a smooth one, the method of differential forms can be extended to possibly singular schemes. This is closely related to Grothendieck's identification between infinitesimal cohomology and de Rham cohomology in characteristic 0, and to Berthelot's rigid cohomology.

Most of results in the paper have parallel versions in the convergent topos, and the proofs of these do not contain surprise. One exception is that the "Theorem B" for quasi-Stein dagger spaces cannot be deduced from the original version by the standard argument (it is a much harder problem). In the special situation we need, such a theorem is made available by Bambozzi [2].

The main result of this note is that the cohomology of the Monsky–Washnitzer topos of a certain natural sheaf agrees with an "analytic cohomology", which in turn agrees with rigid cohomology of Berthelot, see Corollary 5.19. Results about twisted coefficients will appear elsewhere.

Notation. Throughout this paper we fix the following notation. Let k be a field of characteristic $p > 0$. Let V be a complete discrete valuation ring of characteristic 0 with a uniformizer ϖ such that $V/\varpi V = k$. Let K be the fraction field of V, equipped with the standard absolute value $|\cdot|$ such that $|p| = p^{-1}$. Let K^{a} be an algebraic closure of K. For an

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extension L of K with an absolute value $|\cdot|$ extending the one on K, we use $\mathcal{O}_L$ to denote the valuation ring of L, i.e., $\{x \in L : |x| \leq 1\}$. In particular $V = \mathcal{O}_K$.

For each m-tuple of positive numbers $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_i \in |K^\times| \otimes \mathbb{Q}$, we set $|\lambda| = \max \lambda_i$, and

$$\mathfrak{T}_m(\lambda) = \left\{ \sum_{\substack{I = i_1, \ldots, i_m \\ i_1, \ldots i_m \geq 0}} a_I t_1^{i_1} \cdots t_m^{i_m} \in K[\![t_1, \ldots, t_m]\!] : |a_I| \lambda_i^{i_1} \cdots \lambda_n^{i_m} \to 0 \right\}.$$

Then $\mathfrak{T}_m(1, \ldots, 1)$, which will simply be denoted by $\mathfrak{T}_m$, is just the usual Tate algebra $K\langle t_1, \ldots, t_m \rangle$.

If X is a site, *Shv*(X) or $X^\sim$ are used to denote the sheaf topos of X.

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1. Weak formal schemes

We shall review the notion of weakly complete, finitely generated algebras à la Monsky–Washnitzer and the notion of weak formal schemes à la Meredith.

1.1. Let $K\langle t_1, \ldots, t_n \rangle$ be the Tate algebra of restricted power series of n-variables with coefficients in K:

$$K\langle t_1, \ldots, t_n \rangle = \{f \in K[\![t]\!] : |f(a_1, \ldots, a_n)| < \infty, \forall (a_1, \ldots, a_n) \in \mathcal{O}_{K^{\mathrm{a}}}^n\}.$$

Thus $K\langle t_1, \ldots, t_n \rangle$ is the ring of "analytic functions" defined over K, that are convergent on the unit polydisk in K^{a}. We equip $K\langle t_1, \ldots, t_n \rangle$ the p-adic topology. We use $V\langle t_1, \ldots, t_n \rangle$ to denote the ring of power bounded elements (which are precisely power series with coefficients in V) in $K\langle t_1, \ldots, t_n \rangle$. An *affinoid* algebra is a homomorphic image of a Tate algebra. Basic properties of affnoid algebras are recorded nicely in [4]; in our exposition we assume some familiarity with these notions.

We say $f \in K\langle t_1, \ldots, t_n \rangle$ is *overconvergent* if there exists a number $\rho > 1$ (depending on f) such that

$$|f(a_1, \ldots, a_n)| < \infty, \quad \forall (a_1, \ldots, a_n) \in K^n, |a_i| \leq \rho.$$

Clearly all overconvergent elements in $K\langle t_1, \ldots, t_n \rangle$ form dense subring of $K\langle t_1, \ldots, t_n \rangle$, which is called the *Monsky–Washnitzer algebra*, and is denoted by $K\langle t_1, \ldots, t_n \rangle^\dagger$. We also set

$$V\langle t_1, \ldots, t_n \rangle^\dagger = K\langle t_1, \ldots, t_n \rangle^\dagger \cap V\langle t_1, \ldots, t_n \rangle$$

and call it the *integral Monsky–Washnitzer algebra*, and is usually for convenience denoted by $W_m(K)$ or just W_m. A *dagger algebra* is a homomorphic image of a Monsky–Washnitzer algebra.

Dagger algebras admit two different topologies, the topology inherited from the residual norm of a presentation (which is called the *affinoid topology* in the sequel), and a colimit topology. The latter is defined as follows: write $A = W_m/I$, where I is generated by elements in $\mathfrak{T}_m(\rho_0)$ for some $\rho_0 > 1$, then we can write $A = \operatorname{colim}_{1<\rho<\rho_0} \mathfrak{T}_m(\rho)/I\mathfrak{T}_m(\rho)$. Thereby A receives a colimit topology (which is called the "fringe topology" by K. S. Kedlaya) from this presentation. Both the affinoid topology and the colimit topology of A is independent of the choice of the presentation. We will mostly use the affinoid topology, henceforth referred to as *the* topology; when the colimit topology is used we will explicitly specify.

Let A be a dagger algebra. Then its affinoid topology is not complete (i.e., there exists Cauchy sequence that is not convergent). The *completion* $\widehat{A}$ of A can be characterized as follows:

Lemma 1.2. *If $A = W_m/(f_1, \dots, f_r)$ be a (presented) dagger algebra. Then the completion of A is given by $\widehat{A} = K\langle t_1, \dots, t_m\rangle/(f_1, \dots, f_r)$.*

Proof. The algebra $K\langle t_1, \dots, t_m\rangle/(f_1, \dots, f_r)$, being a quotient of the Tate algebra by an ideal (necessarily closed [4, (2.3/8)]), is a complete Banach algebra. The assertion then follows from the fact that the natural map $A \to K\langle t_1, \dots, t_m\rangle/(f_1, \dots, f_r)$ is injective and has dense image. □

We would like to record the universal property of the Monsky–Washnitzer algebra. Recall that an element $a \in A$ in a topological k-algebra is *power bounded* if for any neighborhood U of 1_A, there exists a neighborhood V of 1_A, such that $V \cdot \{a^n : n \in \mathbb{N}\}$ is contained in U. An element $a \in A$ is *weakly power bounded* if λa is power bounded for some $\lambda \in K$, $|\lambda| > 1$. The set of power bounded elements is denoted by A°. The set of weakly power bounded elements in A is denoted by $A^\diamond$. Let A be a dagger algebra with completion $\widehat{A}$, then it is not hard to see that $A^\circ = \widehat{A}^\diamond$.

If A is an affinoid algebra over K, then A° is the subring of A consisting of elements whose supremum seminorm is less than or equal to 1.

Lemma 1.3. *Let R be a dagger K-algebra. Let $r_1, \dots, r_m$ be a collection of power bounded elements in R. Then there exists a unique continuous homomorphism $\mathrm{ev}_{r_1,\dots,r_m} : K\langle t_1, \dots, t_m\rangle^\dagger \to R$ such that $t_i \mapsto r_i$. Thus, in the category of dagger algebras, the functor $R \mapsto (R^\circ)^n$ is represented by W_m.*

Proof. Let A be a dagger K-algebra. Let $0 \to I \to W_m \to A \to 0$ be a presentation of A by a Monsky–Washnitzer algebra. For each $\lambda \in (|K^\times| \otimes \mathbb{Q})^m$, $|\lambda| > 1$, there is an inclusion $\mathfrak{T}_m(\lambda) \to W_m$. Denote by $A(\lambda)$ the image of $\mathfrak{T}_m(\lambda)$ in A. Then $A = \bigcup_{|\lambda|>1} A(\lambda)$.

Let $a \in A$ be a power bounded element. Then a induces a ring homomorphism $\varphi : \mathfrak{T}_1 = K\langle t\rangle \to \widehat{A}$ sending t to a, where $\widehat{A} = \mathfrak{T}_m/I\mathfrak{T}_m$ is the affinoid algebra attached to A. For each $\rho \in |K^\times| \otimes \mathbb{Q}$, $\rho > 1$, we must prove that $\varphi(\mathfrak{T}_1(\rho))$ is contained in $A(\lambda)$ for some suitable λ.

Let $f(t_1, \dots, t_m)$ be a power bounded preimage of a in $W_m \subset \mathfrak{T}_m$. Then there exists λ' such that $f(t_1, \dots, t_n) \in \mathfrak{T}_m(\lambda')$, and is power bounded there. This means that the supremum norm of $f(t_1, \dots, t_n)$ on the polydisk of radius λ' is ≤ 1.

By our choice, the map $\mathfrak{T}_1 \to \widehat{A}$ factors as

$$\mathfrak{T}_1 \xrightarrow{t \mapsto f} \mathfrak{T}_m \to \widehat{A}.$$

Thus it suffices to prove the map $\mathfrak{T}_1 \to \mathfrak{T}_m$ sends $\mathfrak{T}_1(\rho)$ into some $\mathfrak{T}_m(\lambda)$.

Let $g \in \mathfrak{T}_1(\rho)$ be arbitrary. We must prove that $g(f(t_1, \dots, t_m))$ is convergent on some closed disk of radius $|\lambda| > 1$. Since the λ'-Gauß norm of $f(t_1, \dots, t_m)$ is a continuous function in λ', and since the 1-Gauß norm of f is at most 1 (this is the condition that f is power bounded in W_m), it follows that there exists λ such that the λ-Gauß norm of f is at most ρ. It follows that $g(f(t_1, \dots, t_m))$ falls in $\mathfrak{T}_m(\lambda)$: indeed, as supremum norm of $\mathfrak{T}_m(\lambda)$ is the same as the λ-Gauß norm on $\mathfrak{T}_m(\lambda)$, for each point $(c_1, \dots, c_m) \in \mathrm{Sp}(\mathfrak{T}_m(\lambda))$, we have $C = |f(c_1, \dots, c_m)| < \rho$. Thus $|g(f(c_1, \dots, c_m))| = |\sum b_j C^j| < \infty$. This proves the lemma. □

1.4. Weakly complete algebras. Let R be topological V-algebra with ϖ-adic topology. Recall that R is *complete* if $R = \lim_n R/\varpi^n R$ (in particular, we *do* require R to be ϖ-adically separated). In this case, for any $r_1, \dots, r_n \in R$ and any $f \in V\langle t_1, \dots, t_n\rangle$, we can always "evaluate" f at $(r_1, \dots, r_n)$, and get back an element $f(r_1, \dots, r_n) \in R$. We say that (the ϖ-adic topology of) R is *weakly complete*, if it is ϖ-adically separated (therefore the canonical map $R \to \widehat{R}$ is injective), and for any finite collection of elements $r_1, \dots, r_n \in R$, any *overconvergent* power series $f \in V\langle t_1, \dots, t_n\rangle^\dagger$ with coefficients in V, the element $f(r_1, \dots, r_n) \in \widehat{R}$ falls in R.

For example, any ϖ-adically complete V-algebra is weakly complete. The integral Monsky–Washnitzer algebra $V\langle t_1, \dots, t_n\rangle^\dagger$ is weakly complete. The polynomial algebra $V[t_1, \dots, t_n]$ is not weakly complete.

The *weak completion* of a ϖ-adic V-algebra R is the smallest subalgebra $R^\dagger$ of its ϖ-adic completion $\widehat{R}$ that is weakly complete. For example, the weak completion of $V[t_1, \dots, t_n]$ is $V\langle t_1, \dots, t_n\rangle^\dagger$. If R is already weakly complete, then $R = R^\dagger$ by fiat.

We say a subset S of a ϖ-adic V-algebra *weakly generates* R, if for any $r \in R$, there exist finitely many elements $s_1, \dots, s_m \in S$ (m depends on r) and an element $f \in V\langle t_1, \dots, t_m\rangle^\dagger$, such that $r = f(s_1, \dots, s_m)$. We say R is a *weakly complete, finitely generated* V-algebra if R is weakly generated by a finite subset. Hence, weakly complete, finitely generated algebras are precisely the homomorphic images of some integral Monsky–Washnitzer algebras. Therefore, if R is a weakly complete, finitely generated algebra over V, $R[1/\varpi]$ is a dagger K-algebra.

To keep synchronized with the world of formal schemes, we shall say a weakly complete, finitely generated algebra R over V is *admissible* if it is flat over V. This is equivalent to saying R has no ϖ-torsion elements.

1.5. Weak completed tensor product. The category of weakly complete, finitely generated algebras admits tensor products. Let A_1 and A_2 be two weakly complete, finitely generated algebras over a weakly complete, finitely generated algebra B over V. Then we define $A_1 \otimes_B^w A_2$ to be the smallest weakly complete subalgebra of $\widehat{A}_1 \widehat{\otimes}_{\widehat{B}} \widehat{A}_2$ containing the image of $A_1 \otimes_B A_2$.

Lemma 1.6. *Let R be a weakly complete, finitely generated algebra over V. Then the completion of the dagger algebra $R[1/\varpi]$ is $\widehat{R}[1/\varpi]$.*

Proof. Let $R = W_m^\circ/(f_1, \dots, f_r)$ be a presentation of R. Then the ϖ-adic completion of R is $V\langle t_1, \dots, t_m\rangle/(f_1, \dots, f_r)$, since both algebras have the same quotient modulo ϖ^N. Then the present lemma follows from Lemma 1.2. □

Lemma 1.7. *Let R be an admissible weakly complete, finitely generated algebra over V. Then the set of maximal ideals of $R[1/\varpi]$ is in bijective correspondence with R-algebras V' such that*

1. *$R \to V'$ is surjecitve,*
2. *the composition $V \to R \to V'$ is an integral extension, and*
3. *V' has no ϖ-torsion elements.*

In the sequel, a quotient of R or $\widehat{R}$ of the form V' in the statement will be called a *V-rig-point* of R or $\widehat{R}$.

Proof. By [7, Theorem 1.7(2)], the maximal ideals of $R[1/\varpi]$ are in one-to-one correspondence with the maximal ideals of its completion. By Lemma 1.6, the completion is $\widehat{R}[1/\varpi]$. Since R is admissible, so is $\widehat{R}$, since the latter is faithfully flat over R [7, Theorem 1.7(1)]. Since the points of the rigid analytic space $\mathrm{Sp}(\widehat{R}[1/\varpi])$ are in bijective correspondence with V-rig-point of $\widehat{R}$ [4, (8.3/3), (8.3/6)], it suffices to prove that V-rig-point of R and $\widehat{R}$ agree. First, as any finite V-algebra is automatically p-adically complete, a homomorphism $R \to V'$ naturally factors through $\widehat{R}$. So it suffices to prove that for any surjective map $\widehat{\varphi} : \widehat{R} \to V'$, the composition $\varphi : R \to \widehat{R} \to V'$ remains surjective. Lifting to a presentation, it suffices to assume $R = W_m$. Let $c_i \in V'$ be the image of t_i. Since V' is finite over V, any restricted power series of c_i turns to be a polynomial of c_i. This means that c_i generate V' not only topologically, but also algebraically. Hence the composition

$$V[t_1, \ldots, t_m] \to V\langle t_1, \ldots, t_m\rangle \to V'$$

is surjective. This completes the proof. □

Next, we recall the notion of weak formal schemes of Meredith [10].

1.8. Let R be a weakly complete, finitely generated algebra. In this paragraph we, following Meredith, define a locally topologically ringed space, which is denoted by $\mathrm{Spwf}(R)$.

The ambient set of $\mathrm{Spwf}(R)$ is the set of open prime ideals of R. Since R has ϖ-adic topology, an ideal is open if and only if it contains some power of ϖ. This implies that the ambient set of R agrees with $\mathrm{Spec}(R \otimes_V k)$. We give $\mathrm{Spwf}(R)$ the induced Zariski topology.

Let $f \in R$, define the *dagger localization* of R at f to be

$$R\left\langle \frac{1}{f} \right\rangle^{\dagger} = R\langle t\rangle^{\dagger}/(tf - 1).$$

Plainly, the ambient set of $\mathrm{Spwf}(R\langle 1/f\rangle^{\dagger})$ is the same as $\mathrm{Spec}(R[1/f] \otimes_V k)$. For a finitely generated R-module M, define $M\langle 1/f\rangle^{\dagger}$ to be $M \otimes_R R\langle 1/f\rangle^{\dagger}$.

Theorem 1.9 (Meredith)**.** *Let R be a weakly complete, finitely generated algebra over V. Let M be a finitely generated R-module.*

(1) *For any nonzero $f \in R$, $R\langle 1/f\rangle^{\dagger}$ is flat over R.*
(2) *The presheaf* $\mathrm{Spec}(R[1/f] \otimes_V k) \mapsto M\langle 1/f\rangle^{\dagger}$ *on the category of principal open affine subschemes of* $\mathrm{Spec}(R \otimes_V k)$ *is a sheaf-on-a-basis, thus defines a sheaf on* $\mathrm{Spwf}(R)$*, which is denoted by* $\widetilde{M}$.
(3) $\mathrm{H}^i(\mathrm{Spwf}(R), \widetilde{M}) = 0$ *for all* $i > 0$.

Proof. See [10, Theorem 14]. □

Definition 1.10. A *weak formal scheme* over V is a locally topologically ringed space which is locally isomorphic to $(\mathrm{Spwf}(R), \widetilde{R})$ for some weakly complete, finitely generated algebra R over V.

1.11. Weak completion. (i) Let X be a separated V-scheme of finite type. Then we can also define the *weak completion* $X^{\dagger}$ of X along its special fiber $X \otimes_V k$. This will be a weak formal scheme obtained by gluing the weak completion of open affine schemes of X. For example, $\mathbb{A}_V^{n,\dagger} = \mathrm{Spwf}(V\langle t_1, \ldots, t_n\rangle^{\dagger})$.

(ii) Let $\mathfrak{X}$ be a weak formal scheme over V. Using the completion functor $R \mapsto \widehat{R}$, and gluing, we can always define the completion $\widehat{\mathfrak{X}}$ of a weak formal scheme $\mathfrak{X}$, which is a formal scheme over V. There is a canonical morphism of locally and topologically ringed spaces $\widehat{\mathfrak{X}} \to \mathfrak{X}$.

(iii) Let X be a separated finite type V-scheme, with weak completion $X^\dagger$. Then the completion of X along its special fiber is the same as the completion $\widehat{X^\dagger}$ of the weak formal scheme $X^\dagger$.

2. Dagger spaces and generic fibers of weak formal schemes

In this section we discuss how to associate a generic fiber to a weak formal scheme. We then define the specialization functor which can be used to define a crucial notion in this paper, the tubular neighborhood. We then adapt Berthelot's weak fibration theorem in the present context.

2.1. Dagger spaces. Just like formal schemes have rigid analytic spaces as generic fibers, one can define generic fibers for weak formal schemes. These geometric gadgets are known as *dagger spaces*. The foundation of dagger spaces appeared in [7]. Just like rigid analytic spaces, dagger spaces are not genuine ringed spaces, but spaces with Grothendieck topology. The local pieces of a dagger space are dagger algebras, just like affinoid algebras are local pieces of rigid analytic spaces. We shall generally refer the reader to Große-Klönne's paper for basic properties of dagger spaces.

Introducing dagger spaces into the story is needed in computing the "analytic cohomology" of a k-variety. The procedure is as follows. Start with a k-variety X that embeds into the special fiber a smooth weak formal scheme P over V. Then as we shall define shortly, we can take the "tube" $(X)_P$ of X in P, which will be a dagger space over K. The analytic cohomology of X is then the de Rham cohomology of the dagger space $(X)_P$. This depends on the choice of the "frame" P, but as we shall show in §5, the analytic cohomology agrees with the sheaf cohomology of the Monsky–Washnitzer site. In [7, Theorem 5.1(c)], it is shown that the analytic cohomology defined above agrees with Berthelot's rigid cohomology. Combining these results we shall get our main result.

2.2. Associated rigid analytic spaces. Let X be a dagger space. According to [7, Theorem 2.19], there exists a rigid analytic space $\widehat{X}$, called the *completion* of X or *associated rigid analytic space* of X, together with a morphism of locally G-ringed spaces $\widehat{X} \to X$. This morphism is the terminal object in the category of morphisms of G-ringed spaces $Y \to X$ where Y is a rigid analytic space. The local construction is as follows: if $X = \mathrm{Sp}(A)$, then $\widehat{X} = \mathrm{Sp}(\widehat{A})$, where $\widehat{A}$ is the completion of topological ring A (cf. Lemma 1.2).

Example 2.3. Let $X = \mathrm{Sp}(A)$ be an affinoid dagger space. We say an affinoid subspace Y of X is a Weierstrass domain, if $\widehat{Y}$ is a Weierstraß domain of $\widehat{X}$. Hence, Weierstraß subdomains of A are affinoid subdomains defined by the dagger algebras of the form

$$B = A\langle t_1, \dots, t_n\rangle^\dagger/(t_i - f_i : i = 1, 2, \dots n)$$

for some $f_i \in A$. Here are two observations about Weierstraß domains.

1. The ambient set of $\mathrm{Sp}(B)$ is $\{x \in X : |f_i(x)| \le 1\}$. This is because passing to completion does not change the ambient set and the rigid analytic space $\mathrm{Sp}(\widehat{B})$ has the said form.
2. *The map $A \to B$ has dense image*. This is because the quotient of the smaller ring $A[t_1, \dots, t_n]/(t_i - f_i)$ is dense in the completion $\widehat{B}$.

2.4. Generic fiber of weak formal scheme. Any weak formal V-scheme $\mathfrak{X}$ admits a dagger space $\mathfrak{X}_K$ as its "generic fiber". The construction is essentially the same as the Raynaud generic fiber of a formal scheme. We give the construction in steps (cf. [3, §0.2]).

(1) If $\mathfrak{X} = \mathrm{Spwf}(A)$ is an affine weak formal scheme, then $A[1/\varpi] = A \otimes K$ is a dagger algebra (see 1.4). In this case, $\mathfrak{X}_K$ is just the affinoid dagger space $\mathrm{Sp}(A \otimes K)$ (whose ambient set is the set of maximal ideals of $A \otimes K$. As we have argued in Lemma 1.7, the points of $\mathfrak{X}_K$ [which are the same as the points of $\widehat{\mathfrak{X}}_K$ ([7, Theorem 1.7(2)])] are in a bijection with the quotients of A that are integral, finite, flat over V, i.e., *rig-points* of A.

If V' is a rig-point, then $V' \otimes K$ is a finite extension of K, defining a maximal ideal of $\widehat{A} \otimes K$ which in turn determines a unique maximal of $A \otimes K$. Conversely, if K' is a finite extension of K defined by a maximal ideal of $A \otimes K$, the image R of A in K' is an integral flat V-algebra, i.e., a rig point of A (since A and $\widehat{A}$ have the same image on K', by Lemma 1.7).

If $t_1, \ldots, t_n$ are weak generators of the V-algebra A, then their images in K' contained in the valuation ring of K', and are consequently integral over V. The ring V' generated by them is thus finite over V. Since V is henselian, and V' is an integral domain and finite, it follows that V' is a local V-algebra, defining a formal subscheme $\mathrm{Spwf}(V') \subset \mathrm{Spwf}(A)$ supported at a single closed point of $\mathfrak{X}$, called the *specialization* of the point $x \in \mathrm{Sp}(A \otimes K)$ corresponding to V'.

(2) Now suppose that $\mathfrak{X}$ is a weak formal V-scheme. We define $\mathfrak{X}_K$, as a set, to be the set of all closed formal subschemes Z of $\mathfrak{X}$ that are integral, finite, flat over V. The support of such a subscheme Z is a closed point of $\mathfrak{X}$, which will be called the *specialization* of the point $x \in \mathfrak{X}_K$ corresponding to Z. By associating to any point $x \in \mathfrak{X}_K$ its specialization, we get a set-theoretic map

$$\mathrm{sp} : \mathfrak{X}_K \longrightarrow \mathfrak{X}.$$

For all open affine $U = \mathrm{Spwf}(A) \subset \mathfrak{X}$, $\mathrm{sp}^{-1}(U)$ is in bijection with $\mathrm{Sp}(A \otimes K) = U_K$, which has a structure of dagger space. The dagger structure of $\mathfrak{X}_K$ is furnished by the following proposition.

Proposition 2.5 (Cf. [3], Proposition (0.2.3))**.** *Let $\mathfrak{X}$ be a weak formal V-scheme. Then there is a unique structure of dagger space on the set $\mathfrak{X}_K$ defined in* (2.4) *above, such that the following conditions hold.*

(1) *The inverse image of an open subscheme under the map* $\mathrm{sp} : \mathfrak{X}_K \to \mathfrak{X}$ *is an open subspace of $\mathfrak{X}_K$.*
(2) *The inverse image of an open covering under* sp *is an admissible covering for $\mathfrak{X}_K$.*
(3) *For all open affine $U \subset X$, the structure induced by $\mathfrak{X}_K$ on* $\mathrm{sp}^{-1}(U)$ *agree with U_K defined in* (2.4) *above.*

Moreover, the set-theoretic map sp *induces a morphism of ringed topoi*

$$\mathrm{sp} : (Shv(\mathfrak{X}_K), \mathcal{O}_{\mathfrak{X}_K}) \to (\mathfrak{X}^{\sim}_{Zar}, \mathcal{O}_{\mathfrak{X}}).$$

The dagger space $\mathfrak{X}_K$ is called the generic fiber *of $\mathfrak{X}$.*

Proof. The uniqueness is clear. For the existence, we choose an open covering of $\mathfrak{X}$ by affine weak formal schemes $U_i = \mathrm{Spwf}(A_i)$. For each $f \in A_i$ there is an open affine $D(f) \subset U_i$. Regard f as an element in $A_i \otimes K = \Gamma(U_{iK}, \mathcal{O}_{U_{iK}})$, we then have, set-theoretically

$$\mathrm{sp}^{-1}(D(f)) = \{x \in U_{iK} : |f(x)| \geq 1\}. \tag{2.5.1}$$

In fact, if $x \in U_{iK}$ corresponds to the quotient R of A_i, then the point $\mathrm{sp}(x)$ is in $D(f)$ if and only if f is *not* in the maximal ideal of R, which means that $f \in A \otimes K$ is not in the maximal ideal of the valuation ring $V(x) \subset K(x)$. Therefore, we infer that $x \in U_{iK}$ if and only if $|f(x)| \geq 1$. On the other hand, Moreover, we have

$$\Gamma(D(f), \mathcal{O}_{\mathfrak{X}}) = A_i\langle T\rangle^{\dagger}/(fT-1),$$

hence $D(f)_K$ is a dagger space associated to $(A_i \otimes K)\langle T\rangle^{\dagger}/(fT-1)$, which indeed underlies the set $\mathrm{sp}^{-1}(D(f))$.

Since the ambient topological spaces of U_i are noetherian, the open $U_i \cap U_j$ is quasi-compact, hence is a finite union of open subsets of the form $D(f)$, with $f \in A_i$ (resp. $f \in A_j$); it follows that

$$U_{iK} \cap U_{jK} = \bigcup_f D(f)_K$$

is open in U_{iK} (resp. U_{jK}). Fix a finite open cover of $U_i \cap U_j$ by the opens $D(f_\alpha)$, $f_\alpha \in A_i$, and, for all α, a finite open covering $D(f_{\alpha\beta})$ of $D(f_\alpha)$ with $f_{\alpha\beta} \in A_j$. Then $\mathrm{sp}^{-1}(D(f_{\alpha\beta}))$ form a finite open covering of $U_{iK} \cap U_{jK}$ by the special domains of U_{iK} and $D(f_\alpha)_K$; but as $D(f_\alpha)_K$ is a special domain in U_{jK}, $\mathrm{sp}^{-1}(D(f_{\alpha\beta}))$ are also special domains of U_{iK}. Therefore, $\mathrm{sp}^{-1}(D(f_{\alpha\beta}))$ form an admissible covering of $U_{iK} \cap U_{jK}$ in U_{iK} and U_{jK}. It follows that the structures of rigid analytic space on the domains $D(f_{\alpha\beta})$ induced by U_{iK} and U_{jK} are the same. We can thus define a rigid analytic space structure on $\mathfrak{X}_K$.

Recall that saturated subset of a set A with respect to a map $f : A \to B$ are subsets of A of the form $f^{-1}(C)$ for $C \subset B$. In this terminology, the open subsets described above defining $\mathfrak{X}_K$ are saturated with respect to the specialization map $\mathrm{sp} : \mathfrak{X}_K \to \mathfrak{X}$. Therefore to check the condition (1) and (2) we can work locally on $\mathfrak{X}$, hence we can assume that $\mathfrak{X} = \mathrm{Spwf}(A)$. Since the open subsets $D(f)$ form a basis of the topology of $\mathrm{Spwf}(A)$, and since $\mathrm{sp}^{-1}(D(f))$ are indeed admissible open subsets of $\mathrm{Sp}(A)$. This proves (1). (2) is proven similarly. The condition (3) is automatic by our construction.

Finally, by (1) and (2), sp induces a morphism of categories with pretopologies, hence a morphism of sheaf topoi $\mathrm{sp} : \mathit{Shv}(\mathfrak{X}_K) \to \mathfrak{X}^{\sim}_{\mathrm{Zar}}$. If U is an open affine of $\mathfrak{X}$, we have by construction

$$\Gamma(U, \mathrm{sp}_* \mathcal{O}_{\mathfrak{X}_K}) = \Gamma(\mathrm{sp}^{-1}U, \mathcal{O}_{\mathfrak{X}_K}) = \Gamma(U, \mathcal{O}_{\mathfrak{X}}) \otimes K.$$

The presheaf $V \mapsto \Gamma(V, \mathcal{O}_{\mathfrak{X}}) \otimes K$ is a sheaf on U, because the subspace U is noetherian. We therefore get an identification

$$\mathrm{sp}_* \mathcal{O}_{\mathfrak{X}_K} = \mathcal{O}_{\mathfrak{X}} \otimes K,$$

and, for all open subset U, the homomorphism

$$\Gamma(U, \mathcal{O}_{\mathfrak{X}}) \otimes K \to \Gamma(U, \mathrm{sp}_* \mathcal{O}_{\mathfrak{X}_K})$$

is an isomorphism whenever U is quasi-compact (hence noetherian). In particular, the morphism sp is a morphism of ringed topoi. □

2.6. Let $\mathcal{I}$ be a coherent open ideal of a weak formal scheme $\mathfrak{X}$. Let Z be the vanishing scheme of $\mathcal{I}$. The *open tubular neighborhood* of Z in $\mathfrak{X}$, notation $(Z)_{\mathfrak{X}}$, is the subset of $\mathfrak{X}_K$ defined by $\mathrm{sp}^{-1}(Z)$. (As a side comment, we shall not need tubular neighborhoods for locally closed immersions.) A priori the tubular neighborhood is only a subset of $\mathfrak{X}_K$, but we claim $(Z)_{\mathfrak{X}}$ *is an admissible open subspace of* $\mathfrak{X}_K$. The problem being local, so we can assume $\mathfrak{X}$ is affine. Then the next lemma then justifies our assertion.

Lemma 2.7. *Let the notation be as in* 2.6*. Assume that* $\mathfrak{X} = \mathrm{Spwf}(R)$ *is affine,* $\Gamma(\mathfrak{X}, \mathcal{I})$ *is generated by* $f_1, \ldots, f_r \in R$*. Then*

$$\begin{aligned}(Z)_{\mathfrak{X}} &= \{x \in \mathfrak{X}_K : |f_i(x)| < 1\} \\ &= \{x \in \mathfrak{X}_K : |f(x)| < 1, \forall f \in I\}.\end{aligned}$$

is an admissible open subspace of $\mathfrak{X}_K$ *which is a nested union of Weierstraß domains.*

Proof. Since $Z = \mathfrak{X} \setminus \bigcup D(f_i) = \mathfrak{X} \setminus \bigcup_{f \in I} D(f)$, we have the set-theoretic equality $\mathrm{sp}^{-1}(Z) = \bigcap \mathfrak{X}_K \setminus \mathrm{sp}^{-1}(D(f_i)) = \bigcap_{f \in I} \mathfrak{X}_K \setminus \mathrm{sp}^{-1}(D(f))$. Then both equality immediately follows from (2.5.1). Clearly

$$\{x \in \mathfrak{X}_K : |f_i(x)| < 1\} = \bigcup_n \{x \in \mathfrak{X}_K : |f_i(x)| \leq \varepsilon_n\} \tag{2.7.1}$$

where $\varepsilon_n \in |K^\times| \otimes \mathbb{Q}$ is a sequence of numbers converging to 1. Each subspace

$$\{x \in \mathfrak{X}_K : |f_i(x)| \leq \varepsilon_n\}$$

is a Weierstraß domain of $\mathrm{Sp}(R \otimes K)$. This justifies the second assertion. □

Lemma 2.8. *Let the notation be as in* 2.6*. The completion* (*see* 2.2) *of the dagger space* $(Z)_{\mathfrak{X}}$ *equals the rigid analytic tubular neighborhood of* Z *in* $\widehat{\mathfrak{X}}$*.*

Proof. We can assume $\mathfrak{X}$ is affinoid. In the view of (2.7.1), we further reduce ourselves to proving that the dagger Weierstraß domains defined by $|f_i| \leq \varepsilon$ have the rigid analytic Weierstraß domains defined by $|f_i| \leq \varepsilon$ as their completions. This follows from Lemma 1.2. □

Definition 2.9. A *quasi-Stein* dagger space U is a dagger space that can be written as a nested admissible union of affinoid subdomains U_i, each U_i is a Weierstraß domain of U_{i+1}. Let $f : U' \to U$ be a morphism of dagger spaces. We say that f is a *quasi-Stein morphism*, if there exists an admissible open covering of U by affinoid subdomains V_i such that $f^{-1}(V_i)$ are quasi-Stein spaces, i.e., spaces that are nested union of Weierstraß domains.

Corollary 2.10. *Let the notation be as in* 2.6*. The morphism* $(Z)_{\mathfrak{X}} \to \mathfrak{X}_K$ *is a quasi-Stein morphism.*

Example 2.11. Let R be a weakly complete, finitely generated algebra. Let $I = (f_1, \ldots, f_r)$ be an open ideal of R. Then the tubular neighborhood of $Z = \mathrm{Spec}(R/I)$ in $\mathfrak{X} = \mathbb{A}_R^{n,\dagger}$ is the subspace in $\mathrm{Sp}(R \otimes K\langle t_1, \ldots, t_s\rangle^\dagger)$ defined by $|f_i| \leq 1$ and $|t_j| < 1$, that is, the product $(Z)_{\mathfrak{X}} \times D(0; 1^-)^n$. Here, $D(0; 1^-)$ is the rigid analytic unit *open* disk. As the open unit disk is partially proper, there is no need to specify whether we are considering the rigid analytic space or the dagger space, as the two notion are equivalent for partially proper spaces [7, Theorem 2.27].

Next we prove Berthelot's weak fibration theorem in the context of weak formal scheme following Berthelot's strategy.

Lemma 2.12. *Assume there exists a commutative diagram*

$$\begin{array}{ccc} & & \mathfrak{X} \\ & \overset{u}{\nearrow} & \\ Z & & \downarrow f \\ & \underset{v}{\searrow} & \\ & & \mathfrak{Y} \end{array}$$

in which f is a morphism of weak formal schemes, u and v are closed embeddings of Z into the special fibers of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. Assume that the completion $\widehat{f}$ of f is an étale morphism. Then f induces an isomorphism of tubular neighborhoods

$$(Z)_{\mathfrak{X}} \xrightarrow{\sim} (Z)_{\mathfrak{Y}}$$

Proof. By Lemma 2.8, the completion of the tubular neighborhoods $(Z)_{\mathfrak{X}}$ and $(Z)_{\mathfrak{Y}}$ are the rigid analytic tubular neighborhoods $(Z)_{\widehat{\mathfrak{X}}}$ and $(Z)_{\widehat{\mathfrak{Y}}}$. By [3, Proposition 1.3.1], these rigid analytic tubular neighborhood are isomorphic. The lemma then follows from [7, Theorem 2.19(4)], which asserts that a morphism of dagger spaces is an isomorphism if and only if its completion is. □

Lemma 2.13. *Assume there exists a commutative diagram*

$$\begin{array}{ccc} & & \mathfrak{X} \\ & \overset{u}{\nearrow} & \\ Z & & \downarrow f \\ & \underset{v}{\searrow} & \\ & & \mathfrak{Y} \end{array}$$

in which f is a morphism of admissible weak formal schemes, u and v are closed embeddings of Z into the special fibers of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. Assume that the completion $\widehat{f}$ of f is a smooth morphism of formal schemes relative dimension n. Then there exists an admissible open covering $\{V_\alpha\}$ of $(Z)_{\mathfrak{Y}}$ such that, if U_α is the preimage of V_α under the natural morphism $(Z)_{\mathfrak{X}} \to (Z)_{\mathfrak{Y}}$, then there is an V_α-isomorphism

$$U_\alpha \xrightarrow{\sim} V_\alpha \times D(0;1^-)^n$$

of dagger spaces over V_α. Cf. [3, *Théorème* 1.3.2].

Proof. The problem is Zariski local on Z, so we are free to shrink. On the level of formal schemes, one can find a factorization of the morphism $\widehat{f}$

$$\mathfrak{X} \xrightarrow{h} \widehat{\mathbb{A}}^n_V \times \mathfrak{Y} \xrightarrow{\mathrm{pr}_2} \mathfrak{Y},$$

where h is an étale morphism of formal schemes, and Z embeds in the zero section of $\widehat{\mathbb{A}}^n_V \times \mathfrak{Y}$. The second arrow descends to weak formal schemes, but h does not necessarily descend. However, by the Artin approximation theorem for weakly complete, finitely generated algebras, see [13, (2.4.2)], we can find an approximation g of h, which agrees with h on special fibers, and such that g descends to weak formal schemes. If we could show g is étale, then we can apply Lemma 2.12 above and conclude the proof. Note that g reduces to an étale morphism on the special fiber, so in order to prove g itself is étale, we need prove that it is flat. This can be deduced from a suitable version of local criterion of flatness. Let me collect all needed information. By Fulton's theorem, weakly complete, finitely generated algebras are Noetherian, hence they satisfy the Artin-Rees lemma, i.e., of type (APf) defined in [6, 7.4.11 above]. Since the completions of the affine weak formal schemes are formal spectra of complete V-algebras, they are of ϖ-adic Zariski type, see [6, Proposition 7.3.5]. In particular, if $\mathfrak{X} = \mathrm{Spwf}(A)$, $\mathfrak{Y} = \mathrm{Spwf}(B)$, then $\widehat{A}$ is $\varpi(\widehat{B}\langle t_1, \dots t_n\rangle) \cdot \widehat{A}$-Zariski, i.e., $\varpi\widehat{A}$-Zariski. Taking $I = (\varpi)$, we wish to apply [6, Proposition 8.3.8], condition (c), so we need to verify that $\mathrm{Tor}_1^{\widehat{B}\langle t_1,\dots,t_n\rangle}(\widehat{A}, B_0[x_1, \dots, x_n])$ is zero, where $B_0 = \widehat{B}/\varpi$. Since we have assumed our weak formal schemes are admissible, $\widehat{A}$ has no ϖ-torsion. Thereby tensoring the exact sequence

$$0 \to \widehat{B}\langle t_1, \dots, t_n\rangle \xrightarrow{\cdot\varpi} \widehat{B}\langle t_1, \dots, t_n\rangle \to B_0[t_1, \dots, t_n] \to 0$$

with $\widehat{A}$ preserves exactness. This checks the vanishing of the first Tor group. Thereby we have verified all needed conditions to ensure the flatness. This completes the proof. □

3. The Monsky–Washnitzer site

In this section we define the Monsky–Washnitzer site for a variety X over k (relative to V). We also prove some basic functoriality of the topos of this site that are needed in later sections.

Definition 3.1. Let X be a k-variety. A *widening* for X is a pair $(P, z : Z \to X)$ (which is sometimes referred to as P if no confusion is likely), where P is an admissible weak formal scheme, Z a subscheme of P defined by a coherent open ideal, and z a morphism of k-schemes. We say P is *affine* if z is an affine morphism; we say P is absolutely affine if P is affine. To paraphrase, an *absolutely affine widening* T of X consists of

- a morphism of k-schemes $\mathrm{Spec}(R) \to X$,
- a weakly complete, finitely generated, flat V-algebra A, and
- a surjective ring homomorphism $A \to R$.

A *morphism* of widenings $u : (P_1, z_1 : Z_1 \to X) \to (P_2, z_2 : Z_2 \to X)$ is a commutative diagram

$$\begin{array}{ccccc} P_1 & \longleftarrow & Z_1 & \xrightarrow{z_1} & X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_0} & & \| \\ P_2 & \longleftarrow & Z_2 & \xrightarrow[z_2]{} & X \end{array} .$$

When a widening is absolutely affine, we also use the ring-theoretic notation $A \to R \leftarrow \mathcal{O}_X$ to denote it, instead of the geometric notation.

An *enlargement* is a widening (P, Z, z) such that Z agrees with $P \otimes_V k$. Thus an absolutely affine widening $T = (A \to R \leftarrow \mathcal{O}_X)$ is an *absolutely affine enlargement* of X if $R = A/\varpi A$.

3.2. Remark on terminologies. In Ogus's paper [12], a *widening* is a pair $(P, z : Z \to X)$, where P is a flat formal scheme over V with Z as a subscheme of definition (i.e., P is $\mathcal{I}_Z$-adically complete). A pair $(P, z : Z \to X)$ with P admissible and $Z \subset P \otimes_V k$ is known as a *prewidening* (following Shiho [14]). In the convergent topos, a prewidening and its corresponding widening represent the same sheaf. From this perspective, our widening seems to be better called a prewidening, as weak formal schemes are always ϖ-adic. But it does not seem to be very useful to further define weak formal schemes that are not ϖ-adic, so we shall stick with the current terminology.

One attempts to use widenings to define a site. But there is no reasonable meaning of fiber products of widenings. This is caused by the presence of ϖ-torsions in the fiber product of weakly complete, finitely generated algebras.

Example 3.3 (No obvious fiber product for widenings)**.** Consider the following diagram of widenings of $X = \operatorname{Spec}(k)$. (For the ease of thinking, I also put the "rigid analytic" counter part of the picture on the left.)

$$\begin{array}{ccccc}
K\langle s\rangle^\dagger & \longleftarrow & V\langle s\rangle^\dagger & \xrightarrow{s,\varpi\mapsto 0} & k \\
{\scriptstyle t\mapsto\varpi}\uparrow & & {\scriptstyle t\mapsto\varpi}\uparrow & & \| \\
K\langle t,s\rangle^\dagger & \longleftarrow & V\langle t,s\rangle^\dagger & \xrightarrow[\varpi\mapsto 0]{t,s\mapsto 0} & k \\
{\scriptstyle t\mapsto 0}\downarrow & & {\scriptstyle t\mapsto 0}\downarrow & & \| \\
K\langle s\rangle^\dagger & \longleftarrow & V\langle s\rangle^\dagger & \xrightarrow[s,\varpi\mapsto 0]{} & k
\end{array}$$

In this case, the completed tensor product of the left column is zero. Geometrically, the two homomorphisms correspond to $s \mapsto (s, \varpi)$ and $s \mapsto (s, 0)$, two disjoint embeddings of the closed disk into the 2-dimensional polydisk. However, the fiber product of the special fiber is nonzero. Note that the fiber product of the middle column equals $V[s]/\varpi$, which is ϖ-torsion. Trying to flatten it will result the zero ring.

3.4. Absolute product exists. Nevertheless, the notion of absolute product of widenings can be defined as follows. If $T_1 = (A_1, R_1)$ and $T_2 = (A_2, R_2)$, then we define $T_1 \times T_2 = (A_1 \otimes_V^w A_2, R_1 \otimes_R R_2)$, where $\operatorname{Spec}(R)$ is an open affine scheme on which the morphisms

$$\operatorname{Spec}(R_i) \to X$$

factor into.

Lemma 3.5. *Let R be a finite type k-algebra. Assume that we are given*

- *flat, ϖ-adic, ϖ-adically complete V-algebras A, A_1 and A_2, such that $A/\varpi A = A_i/\varpi A_i = R$;*
- *ring homomorphisms $\varphi_i : A \to A_i$, such that φ_i modulo ϖ agrees with the identity homomorphism* $\mathrm{Id} : R \to R$*.*

Then $A_1 \widehat{\otimes}_A A_2$ is flat over V.

Proof. Set $A_i^{(n)} = A_i/\varpi^n A_i$, and similarly set $A^{(n)} = A/\varpi^n A$. Then the completed tensor product is the inverse limit of $B_n = A_1^{(n)} \otimes_{A^{(n)}} A_2^{(n)}$. Any ϖ-power torsion of the completed tensor product B would then give rise elements in $b_n \in B_n$, for n-sufficiently large, that is killed by ϖ^r. But $\varpi^r \in V/\varpi^n V$, and b_n becomes a torsion element. So B_n would not be flat. Therefore, it thus suffices to prove $B_n = A_1^{(n)} \otimes_{A^{(n)}} A_2^{(n)}$ is flat over V/ϖ^n. But this follows from a suitable version of local criterion of flatness [5, p132, Lemma 5.21(3)]. □

Corollary 3.6. *Let $T = (A, R)$, $T_1 = (A_1, R_1)$ and $T_2 = (A_2, R_2)$ be three enlargements. Assume we are given morphisms $T_1 \to T$ and $T_2 \to T$ that induces open immersions on* *special fibers. Then*

$$(A_1 \otimes_A^w A_2, R_1 \otimes_R R_2)$$

has a natural structure of an absolutely affine enlargement and represents the fiber product $T_1 \times_T T_2$ *in the category of absolutely affine enlargements.*

Proof. Let $\widehat{B}$ be the ϖ-adic completion of a weakly complete, finitely generated algebra B. Let $S = R_1 \otimes_R R_2$. Then T, T_1 and T_2 naturally define enlargements with special fiber $\mathrm{Spec}(S)$, by restricting to the corresponding Zariski localization. By Lemma 3.5, the completed tensor product $\widehat{A}_1 \widehat{\otimes}_{\widehat{A}} \widehat{A}_2$ is flat over V. As the weakly completed tensor product is a subring of the completed tensor product, the weakly completed tensor product is ϖ-torsion free, i.e., flat, as well. ◻

Definition 3.7. Let $\mathrm{Enl}(X/V)$ be the category whose objects are absolutely affine enlargements (A, R). By Corollary 3.6, $\mathrm{Enl}(X/V)$ admits fiber products. We say a collection of morphisms of enlargements $\{T_i \to T\}_{i \in I}$ is a *covering* if the special fibers $T_i \otimes_V k$ form an open covering of the special fiber of T. Endowing with this notion of covering $\mathrm{Enl}(X/V)$ becomes a category with a pretopology. We use $(X/V)_{\mathrm{MW}}$ to denote its sheaf topos, and call it the *Monsky–Washnitzer topos* of X/V.

Example 3.8. We give a few examples of sheaves on the Monsky–Washnitzer topos of a k-variety X.

(i) The presheaf which assigns to each (absolutely affine) enlargement $T = (A \to R \leftarrow \mathcal{O}_X)$ the weakly complete, finitely generated algebra A is a sheaf. It is denoted by $\mathcal{O}_{X/V}$.

(ii) The presheaf which assigns to T the dagger algebar $A[1/\varpi]$ is also a sheaf, which is denoted by $\mathcal{O}^{\mathrm{an}}_{X/V}$. We define the *Monsky–Washnitzer cohomology* of X/K to be the sheaf cohomology of $\mathcal{O}^{\mathrm{an}}_{X/V}$:

$$\mathrm{H}^{\bullet}_{\mathrm{MW}}(X/K) = \mathrm{H}^{\bullet}((X/V)_{\mathrm{MW}}, \mathcal{O}^{\mathrm{an}}_{X/V})$$

Note that since $(X/V)_{\mathrm{MW}}$ is not noetherian, the Monsky–Washnitzer cohomology is different from the sheaf cohomology of $\mathcal{O}_{X/V}$ tensored with K. For smooth affine varieties over k, we shall see below that the present definition agrees with the classical one.

(iii) Let P be a widening. Then P defines a sheaf on the Monsky–Washnitzer site which sends an absolutely affine enlargement T to the set $\mathrm{Hom}(T, P)$.

The role of twisted coefficients in a Monsky–Washnitzer topos are played by the so-called Monsky–Washnitzer isocrystals, as defined below.

Definition 3.9 (Crystalline sheaves)**.** A *crystalline* sheaf $\mathcal{F}$ of $\mathcal{O}_{X/V}$-modules (or $\mathcal{O}^{\mathrm{an}}_{X/V}$-modules) is an $\mathcal{O}_{X/V}$-module (or an $\mathcal{O}^{\mathrm{an}}_{X/V}$-module) in the topos $(X/V)_{\mathrm{MW}}$ such that for any morphism $g : T' \to T$ of enlargements, the natural morphism

$$g^* \mathcal{F}_T \to \mathcal{F}_{T'}$$

of Zariski sheaves is an isomorphism.

We say a sheaf $\mathcal{F}$ of $\mathcal{O}_{X/V}$-modules (resp. $\mathcal{O}^{\mathrm{an}}_{X/V}$-modules) is *coherent*, if for each enlargement T, the Zariski sheaf $\mathcal{F}_T$ is coherent (resp. isocoherent). Coherent crystalline $\mathcal{O}_{X/V}$-modules are called *Monsky–Washnitzer crystals*, coherent crystalline $\mathcal{O}^{\mathrm{an}}_{X/V}$-modules are called *Monsky–Washnitzer isocrystals*.

Example 3.10. Let P be a weak formal scheme over V whose completion is smooth over V. Let X be a closed subvariety of $P \otimes_V k$. Then (P, X, Id) is a widening.

Lemma 3.11. *Let the notation be as in Example* 3.10*. Let* (T, Z, z) *be an absolutely affine enlargement, then there is a morphism of widenings* $T \to P$*.*

Proof. Let $\widehat{T}$ be the completion of T. Then Z is a scheme of definition of $\widehat{T}$. Since $\widehat{P}$ is formally smooth, there exists a lift $\widehat{u} : \widehat{T} \to \widehat{P}$ that reduces to $z : Z \to P$. By the Artin approximation theorem of weakly complete, finitely generated algebras [13, (2.4.2)] we get a morphism $u : T \to P$ that approximates $\widehat{u}$. In particular we can arrange u to agree with $\widehat{u}$ on the special fiber. □

The following is a translation of the above result to abstract nonsense.

Corollary 3.12. *Let the notation be as above. Let* h_P *be the sheaf represented by the widening* P *in the Monsky–Washnitzer topos. Then object* h_P *is a covering to the terminal object.*

3.13. For a k-variety X that is not a closed subscheme of a weak formal scheme whose completion is smooth, we can use a variant of the above construction to get a covering of the terminal object of the Monsky–Washnitzer topos, as follows. Let $\{U_i\}$ be a covering of X by affine open subschemes. Let P_i be a weak affine space containing U_i as a closed subscheme of its special fiber. Then (P_i, U_i) is a widening for X/V. Now view each (P_i, U_i) as a sheaf on the Monsky–Washnitzer site, then the collection of sheaves $\{(P_i, U_i)\}$ is a covering of the terminal object of the Monksy-Washnitzer topos. The proof is precisely as above (but easier, as we can just use the universal property of the integral Monsky–Washnitzer algebra, and need not to resort to the Artin approximation).

We close this section by proving the functoriality of Monsky–Washnitzer topoi. Note that if $f : X \to X'$ is a morphism of k-varieties. Then f in general does not pull back enlargements. So we don't always have continuous functors on the level of categories with pretopologies. Nevertheless f does induce a morphism of topoi $f_{\mathrm{MW}} : (X/V)_{\mathrm{MW}} \to (X'/V)_{\mathrm{MW}}$ since we have a naturally defined "cocontinuous functor". Let's review this notion first.

Definition 3.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with pretopologies. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. The functor u is called *cocontinuous* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every covering $\{V_j \to u(U)\}_{j \in J}$ of $\mathcal{D}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i \in I}$ refines the covering $\{V_j \to u(U)\}_{j \in J}$.

The following lemma tells us how the notion of cocontinuous functors are related to morphisms of sheaf topoi. Recall that if $u : \mathcal{C} \to \mathcal{D}$ is a functor of categories, then

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

is the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. This functor has both left and right adjoints which we denote by u_p and ${}_pu$ respectively. For their definitions, see [SP], Tag `00XF`.

Lemma 3.15. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories with pretopologies. Let* $u : \mathcal{C} \to \mathcal{D}$ *be cocontinuous. Then the* $g_* = {}_pu$ *takes sheaves to sheaves. The pair* g_* *and* $g^{-1} = (u^p)^\#$ (# *stands for the sheafification*) *define a morphism of topoi* g *from* $Shv(\mathcal{C})$ *to* $Shv(\mathcal{D})$*.*

Proof. See [SP], Tag `00XO`. □

Lemma 3.16. *Let* $f : X \to X'$ *be a morphism of* k*-varieties. Then it induces a canonical morphism of the Monsky–Washnitzer topoi*: $f_{\mathrm{MW}} : (X/V)_{\mathrm{MW}} \to (X'/V)_{\mathrm{MW}}$.

Proof. Let $S = (\mathfrak{Z}, Z, z)$ be an enlargement for X/V. Then $(\mathfrak{Z}, Z, f \circ z)$ is an enlargement for X'/V. This defines a functor $u : \mathrm{Enl}(X/V) \to \mathrm{Enl}(X'/V)$. We shall verify that u is cocontinuous in the sense of Definition 3.14. Let $\{T_i = (\mathfrak{V}_i, V_i, v_i)\}$ be a collection of enlargements of X' that covers $u(S)$. Since we are using Zariski topology, this means that $\{\mathfrak{V}_i\}$ is a Zariski open covering for $\mathfrak{Z}$ and $\{V_i\}$ is a Zariski open covering of Z. The morphism v_i thus factors as

$$V_i \hookrightarrow Z \xrightarrow{z} X \xrightarrow{f} X'.$$

In other words, T_i are naturally enlargements for X/V. Therefore the defining property of a cocontinuous functor is satisfied for u, even without refining the covering. This completes the proof. □

4. Universal enlargements

In this section we define the crucial tool in the of the study of the Monsky–Washnitzer topos, the so-called *universal enlargement* construction. In the context of convergent topos this is due to Ogus. In the rigid-analytic context this is Berthelot's tubular neighborhood construction.

4.1. Admissible weak formal blowup. Let A be an admissible (i.e., flat) weakly complete, finitely generated algebra. Let $X = \mathrm{Spwf}(A)$. Let I be an ideal (necessarily coherent) of A. Denote $A_n = A/\varpi^{n+1}A$, and $I_n = \frac{I}{\varpi^{n+1}A \cap I}$. Since A has ϖ-adic topology, I is *open* if and only if it contains some power of ϖ. Assume that I is open, then for each n, define

$$X_{I_n} = \mathrm{Bl}_{I_n}(\mathrm{Spec}(A_n)) = \mathrm{Proj}_X\left(\oplus_{j=0}^{\infty} I_n^j\right).$$

There is a natural map $X_{I_n} \to \mathrm{Spec}(A_n)$, and by passing to limits, a morphism of formal schemes

$$\pi : \widehat{X}_I \to \widehat{X}$$

where $\widehat{X}_I = \mathrm{colim}\, X_{I_n}$ is called the *admissible formal blowup* of $\widehat{X} = \mathrm{Spf}(\widehat{A})$ along I. If instead of performing completion we perform the weak completion, we get a weak formal scheme X_I.

Now let X be an admissible weak formal scheme with a covering of admissible affine weak formal schemes $\{U_i = \mathrm{Spwf}(A_i)\}$. Let $\mathcal{A}$ be an open coherent sheaf of ideals on X. Then $\mathcal{A}|_{U_i}$ is an open coherent $\mathfrak{a}_i$ of A_i. Then we can define the admissible weak formal blowups $U_{i,\mathfrak{a}_i}$. It can be shown admissible formal blowups satisfy a certain universal property similar to that of the usual blowup, so we can glue the formal schemes $U_{i,\mathfrak{a}_i}$ together, and obtain another weak formal scheme $X_{\mathcal{A}}$. This is called the *admissible weak formal blowup* of X along $\mathcal{A}$.

Theorem 4.2. *Let X be an admissible weak formal $\mathfrak{o}$-scheme. Let $\mathcal{A}$ be a sheaf of coherent open ideal. Then $X_{\mathcal{A}}$ is an admissible weak formal scheme over V.*

Proof. By [4, (8.2/8)], the completion of X is ϖ-torsion free over V. Therefore $\mathcal{O}_X$, being a subalgebra of $\mathcal{O}_{\widehat{X}}$, is also ϖ-torsion free. Since V is a valuation ring, being ϖ-torsion free is the same as being flat. □

Definition 4.3. Let X be an admissible weak formal scheme. Let $\mathcal{I}$ be a coherent open ideal of $\mathcal{O}_X$. In the sequel, let $T_{m,\mathcal{I}}(X)$ be the open formal subscheme of the admissible blowup $X_{\varpi+\mathcal{I}^m}$ (which is the weak completion of the scheme $\mathrm{Proj}_X\big(\oplus_{j=0}^{\infty}\{(\varpi) + \mathcal{I}^m\}^j\big)$)

defined by the nonvanishing locus of the Cartier divisor ϖ. Thus, the weak formal scheme $T_{m,\mathcal{I}}(X)$ is admissible, thanks to Theorem 4.2, and there exists a natural morphism of weak formal schemes over $\mathfrak{o}$:

$$\tau_m : T_{m,\mathcal{I}}(X) \to X$$

such that the ideal $(\varpi + \mathcal{I}^m) \cdot \mathcal{O}_{T_{m,\mathcal{I}}(X)}$ generated by the inverse image of ϖ and elements in $\mathcal{I}^m$ is the principal ideal $\varpi \cdot \mathcal{O}_{T_{m,\mathcal{I}}(X)}$.

Example 4.4. Let $A = V\langle x\rangle^\dagger$. Then A is an admissible weakly complete, finitely generated V-algebra. Let $X = \operatorname{Spwf}(A)$, and $\mathcal{I} = (x)$. Then

$$T_{n,\mathcal{I}}(X) = \operatorname{Spwf}\left(V\langle x, s\rangle^\dagger/(x^n - \varpi s)\right).$$

The generic fiber $T_{n,\mathcal{I}}(X)_K$ of $T_{n,\mathcal{I}}(X)$ is the closed overconvergent disk $\operatorname{Sp}(K\langle x\rangle^\dagger)$ in K of radius $|\varpi|^{1/n}$. When $n \to \infty$, the union of these closed (overconvergent) disks is the open disk of radius 1.

Lemma 4.5. *Let the notation be as in Definition* 4.3. *Suppose $n_1 > n_2$ are non-negative integers. Then there exists a unique commutative diagram*

$$\begin{array}{ccc} T_{n_2,\mathcal{I}}(X) & \xrightarrow{\tau_{n_1,n_2}} & T_{n_1,\mathcal{I}}(X) \\ & {\scriptstyle \tau_{n_2}}\searrow \quad \swarrow{\scriptstyle \tau_{n_1}} & \\ & X & \end{array}.$$

Thus $\{T_{n,\mathcal{I}}(X)\}$ is naturally an ind-weak formal scheme.

Proof. The arrows exist even before we pass to the weak formal completion, as shown by the following. By the universal property of blowup, it suffices to prove the ideal in $\mathcal{O}_{T_{n_2,\mathcal{I}}(X)}$ generated by $\tau_{n_2}^{-1}((\varpi) + I^{n_1})$ is the principal ideal (ϖ). It is clear that ϖ is in the ideal generated by $((p) + I^{n_1})$. Let $f_1 \cdot f_{n_1}$ be local sections opf $\mathcal{I}^{n_1}$. On (the uncompleted version of) $T_{n_2,\mathcal{I}}(X)$, the open piece of a certain Proj construction obtained by inverting a degree 1 section ϖ of $\mathcal{O}(1)$, the element

$$g = \frac{f_{i_1} \cdots f_{i_{n_2}}}{\varpi}$$

is a well-defined local *function* on $T_{n_2,\mathcal{I}}(X)$. Since $n_1 > n_2$, we can write

$$f_1 \cdots f_{n_1} = g \cdot f_{n_2+1} \cdots f_{n_1} \cdot \varpi$$

on an open formal subscheme of the un-completed version of $T_{n_2,\mathcal{I}}(X)$. Clearly the expression falls in the ideal generated by ϖ. We win. □

Lemma 4.6. *Let the notation be as in Definition* 4.3. *Assume that X is an affine weak formal scheme. Suppose $n_1 > n_2$ are non-negative integers. Then the morphism on generic fibers*

$$\tau_{n_1,n_2,K} : T_{n_2,\mathcal{I}}(X)_K \to T_{n_1,\mathcal{I}}(X)_K$$

(*induced by τ_{n_1,n_2} defined in Lemma* 4.5) *is an admissible open immersion of affinoid dagger spaces represented by a Weierstraß subdomain.*

Proof. Clearly, if $U_1 \subset U_2$ are two affinoid subdomains of an affinoid dagger space U, and if U_i are all Weierstraß domains of U, then U_1 is a Weierstraß domain of U_2. Therefore, it suffices to prove that the morphism

$$\tau_{n,K} : T_{n,\mathcal{I}}(X)_K \to X_K$$

is an open immersion of a Weierstraß subdomain. But then according to the definition, the former is obtained by forcing the affinoid functions g/ϖ on X_K to be ≤ 1, where $g \in \mathcal{I}^n$, hence is indeed a Weierstraß domain. We win. □

Lemma 4.7. *Let hypotheses and the notation be as in Lemma* 4.6. *Then the morphism* $\tau^*_{n_1,n_2,K}$ *on affinoid algebras has dense image* (*with respect to the affinoid topology*).

Proof. This is a corollary of Lemma 4.6, because that whenever a morphism $\varphi : U' \to U$ is an open immersion of affinoid dagger spaces that is a Weierstraß domain, φ^* always has dense image: if B is a dagger algebra and $h \in B$, then we can always approximate any element in $B\langle\zeta\rangle^\dagger/(\zeta - h)$ by elements of the form $\sum_{j=1}^N b_j h^j$, which lies in B. □

Lemma 4.8. *Let the notation be as in Definition* 4.3. *We have*

$$\bigcup_{m=1}^{\infty} T_{m,\mathcal{I}}(X)_K = (Z)_X$$

where Z is the vanishing scheme of $\mathcal{I}$. The above covering is also an admissible covering.

Proof. This is a local problem. Then we can write down a presentation of $\mathcal{I}$, and use the explicit description of tubular neighborhood as well as the generic fibers of $T_{n,\mathcal{I}}$. In fact, after inverting ϖ, the mth universal enlargement becomes the domain defined, in X_K, by $|f| \leq |\varpi|$, f runs through elements in $\mathcal{I}^m$, and $(Z)_X$ is defined by $|g| < 1$, for $g \in \mathcal{I}$. The assertion is now clear. □

4.9. Let the notation be as in Definition 4.3. Let Z be the vanishing scheme of Z. Write $T_n = T_{n,\mathcal{I}}(X)$. Recall by definition we have natural morphisms $\gamma_n : T_n \to P$. We have set $Z_n = \gamma_n^{-1}(Z)$, and we have $(T_n \otimes_V k)^{\mathrm{red}} = Z_n^{\mathrm{red}}$. So the ambient topological space of T_n is no different from that of Z_n, which maps to the ambient space of Z in a continuous fashion. So we get a morphism of sheaf topoi: $\gamma_n : T_n^\sim \to Z_{\mathrm{Zar}}^\sim$. This is not a morphism of ringed topoi though.

We define a site $\vec{T}$ as follows. The objects are the open subsets of some T_n. If U_n is open in T_n and U_m is open in T_m, with $n \leq m$, then a morphism $U_n \to U_m$ is a commutative diagram

$$\begin{array}{ccc} U_n & \longrightarrow & U_m \\ \downarrow & & \downarrow \\ T_n & \longrightarrow & T_m \end{array} .$$

If $n > m$ there is no morphism from U_n to U_m. The coverings of U_n are usual Zariski coverings.

We can similarly define a site $\vec{T}_K$ for the generic fibers $T_{n,K}$. The specialization morphisms give rise to a morphism of sites $\vec{T}_K \to \vec{T}$ which is denoted by $\vec{\mathrm{sp}}$. Since $T_{n,K}$ form an admissible nested covering of $(Z)_X$, there is a natural full and faithful morphism $(Z)_X^\sim \to \vec{T}_K^\sim$ of sheaf topoi, given by restrictions. Putting everything together, we get a commutative diagram of topoi:

(4.9.1)
$$\begin{array}{ccccc} \vec{T}_K^\sim & \xleftarrow{\mathrm{res}} & (Z)_X^\sim & \longrightarrow & X_K \\ \downarrow{\scriptstyle \vec{\mathrm{sp}}} & & \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{sp}} \\ \vec{T}^\sim & \xrightarrow{\gamma} & Z_{\mathrm{Zar}}^\sim & \longrightarrow & X_{\mathrm{Zar}}^\sim \end{array}$$

where $\gamma_*(F_n)$ is the inverse limit of the sheaves $\gamma_{n*}(F_n)$. If (F_n) is a collection of coherent sheaves on $\vec{T}_K$, then $\mathrm{R}^i\mathrm{sp}_*(F_n) = 0$, since cohomology of coherent sheaves on an affinoid dagger space is zero [7, Proposition 3.1] for all $i > 0$. By definition, $R^i\mathrm{res}_*(F)$ is zero for any sheaf on $(Z)_X$.

Definition 4.10. Let the notation be as in 4.9. Let $\mathcal{O}_{\vec{T}_K} = (\mathcal{O}_{T_{n,K}})$ be the sheaf on $\vec{T}_K$ defined by $\mathrm{res}_*(\mathcal{O}_{(Z)_X})$. We say a sheaf (F_n) of $\mathcal{O}_{\vec{T}_K}$-modules is *good* if

(1) F_n is coherent over $\mathcal{O}_{T_{n,K}}$,
(2) Zariski locally on Z, the transition maps $F_n \otimes \mathcal{O}_{T_{n,K}} \to F_{n-1}$ are isomorphism.

Then good sheaves are essential images of coherent $\mathcal{O}_{(Z)_X}$-modules under the functor res_*. We say a sheaf on $\vec{T}$ is good if it is an essential image of a good sheaf on $\vec{T}_K$ under sp.

We next need an incarnation of Kiehl's Theorem B for dagger spaces. Note that Theorem B for dagger spaces is significantly harder than the original. The simple-minded method of reducing to completion does not work, as one does not have control of the radii of the fringe algebras appeared in the process. Luckily, the theorem is made available by F. Bambozzi [2] for quasi-Stein dagger spaces that are closed subspaces of products of dagger disks and open disks.

Lemma 4.11. *Let (F_n) be a good sheaf on $\vec{T}$. Then*

(1) $R\gamma_*(F_n) = \gamma_*(F_n)$.
(2) *For any coherent sheaf $\mathcal{F}$ on the dagger space $(Z)_X$, we have $R\mathrm{sp}_*(\mathcal{F}) = \mathrm{sp}(\mathcal{F})$.*

Proof. Since the problem is local, we are reduced to the situation where Z and X are affine.

We first show that the first assertion follows from the second. Each F_n is of the form $G_n \otimes K$ for some coherent sheaf. Since T_n has Zariski topology, we have $\mathrm{H}^e(T_n, F_n) = \mathrm{H}^e(T_n, G_n) \otimes K$ The latter vanishes for $e > 0$. This shows that the functor $\vec{\mathrm{sp}}$ is exact on good sheaves, or $\vec{\mathrm{sp}}_* = R\vec{\mathrm{sp}}_*$. Since $R\gamma_* \circ R\vec{\mathrm{sp}}_* \circ R\mathrm{res}_* = R\mathrm{sp}_*$ the vanishing of $R^i\mathrm{sp}$, $R^j\vec{\mathrm{sp}}$ and $R^\ell\mathrm{res}_*$ $(i, j, \ell > 0)$ imply the vanishing of $R^s\gamma_*$ for all $s > 0$ by the spectral sequence of composition of functors.

Now we turn to the proof of (2). With the hypothesis that Z and X are affine, $(Z)_X$ is a closed subspace of a quasi-Stein space satisfying the hypotheses of [2, Corollary 4.22]. In fact, X_K is a closed subspace of a dagger closed disk defined by a collection of function, $(Z)_X$ is the closed subspace of $X_K \times D(0; 1^-)^r$ defined by the equations of $t_i = f_i$, where $f_1, \dots, f_r$ are a collection of generators of the ideal defining Z. Thereby by loc. cit. the cohomology groups $\mathrm{H}^i((Z)_X, \mathcal{F})$ are zero, i.e., "Theorem B holds". The exactness of res_*, the vanishing of coherent cohomology of affinod dagger spaces (whence the vanishing of $R^i\mathrm{sp}_*$ for the specialization morphisms $\mathrm{sp} : T_{n,K} \to T_n$), and the commutativity of the diagram (4.9.1) together imply the lemma. □

Definition 4.12. Given a pair (X, Z) as in 4.9, we define the *analytic cohomology* of Z with respect to X to be the de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}((Z)_X)$. To be precise, on any dagger space Y there is a complex of coherent sheaves $\Omega^\bullet_Y$ defined as usual by using exterior powers of the *weakly completed* 1-differentials on Y, and the de Rham cohomology of Y is then the hypercohomology of this complex. (We have avoided the cumbersome dagger notation, as henceforth we consider only weakly completed and completer differentials. We shall put hats on completed ones.)

Definition 4.13. Let X now be a variety over k. Let (P, Z, z) be a widening of X. Then the construction in 4.9 gives rise a collection of enlargements

$$(T_{n,I_Z}(X), Z_n, z_n : Z_n \to Z \to X).$$

The ind-object $\{(T_{n,Z}(X), Z_n, z_n : Z_n \to Z \to X)\}$ in the Monsky–Washnitzer site is called the universal enlargements associated to the widening P.

4.14. It is not hard to see that the colimit of $\{(T_{n,Z}(X), Z_n, z_n : Z_n \to Z \to X)\}$ (in the Monsky–Washnitzer topos) surjects onto the sheaf represented by the widening P. To check that is an isomorphism, it suffices to show the transition maps are injective as sheaves. Checking locally, this is implied by the density result of the Weierstraß domains. Thereby the colimit of $\{(T_{n,Z}(X), Z_n, z_n : Z_n \to Z \to X)\}$ (in the Monsky–Washnitzer topos) is isomorphic to the sheaf represented by P.

5. Computing Monsky–Washnitzer cohomology

Overview. In this section we prove that we can use analytic cohomology to compute Monsky–Washnitzer cohomology. To begin with, we draw a big diagram of topoi indicating what will be going on. In the sequel, let X be a variety over k. Let P be a smooth weak formal scheme containing X as a subvariety. Therefore (P, X, Id) is a widening, which also defines a sheaf on the Monsky–Washnitzer site. We use $(X/V)_{\mathrm{MW}}|_P$ to denote the relative site consisting of enlargements of (X/V) together with a specified arrow (of sheaves) to the widening P.

$$\begin{array}{ccccc}
(X/V)_{\mathrm{MW}}|_P^{\sim} & \xrightarrow{\varphi_*} & \vec{P}^{\sim} & \xleftarrow{\vec{\mathrm{sp}}} & \vec{P}_K^{\sim} \\
\downarrow{\scriptstyle j_P} & & \downarrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \mathrm{res}} \\
(X/V)_{\mathrm{MW}}^{\sim} & \xrightarrow[u_{X/V}]{} & X^{\sim} & \xleftarrow[\mathrm{sp}]{} & (X)_P^{\sim}
\end{array}$$

We need the whole section to explain the details of these functors, but let us at this point have an overview of the entire process. The left diagram is about analytic cohomology. The analytic cohomology is computed by $R\Gamma(X, R\mathrm{sp}_*\Omega^\bullet_{(X)_P})$. By the "Theorem B" of Bambozzi–Kiehl, Lemma 4.11, the specialization functor is acyclic with respect to coherent sheaves, so we can suppress the derived symbol and use the following detour to compute the analytic cohomology:

$$R\Gamma(X, \mathrm{sp}_*\Omega^\bullet_{(X)_P}) = R\Gamma(X, (\gamma_* \circ \vec{\mathrm{sp}}_* \circ \mathrm{res}_*)\Omega^\bullet_{(X)_P}).$$

We have also explained that the all three functors $\vec{\mathrm{sp}}$, res, and γ are of vanishing higher direct images in Lemma 4.11.

On the left square, the functor φ_* is not part of a morphism of topoi. But it is exact. On the Monsky–Washnitzer topos, we can construct, using differential forms, a complex of sheaves which we call dR_P. It turns out $Ru_{X/V,*}\mathrm{dR}^\bullet_P = \mathrm{sp}_*\Omega^\bullet_{(X)_P}$. So the analytic cohomology can also be computed by

$$\mathrm{R}\Gamma(X, Ru_{X/V,*}\mathrm{dR}^\bullet_P).$$

It turns out $Rj_{P,*} = j_{P,*}$ for the above complex, and there is a so-called "Poincaré lemma", i.e., there is a quasi-isomorphism

$$\mathcal{O}^{\mathrm{an}}_{X/V} \to \mathrm{dR}^\bullet_P$$

Thereby we see the above complex can be replaced by

$$R\Gamma(X, Ru_{X/V,*}\mathcal{O}^{\mathrm{an}}_{X/V}),$$

that is, *the Monsky–Washnitzer cohomology agrees with the analytic cohomology*.

Let us carry out the above scheme of proof.

5.1. In this section, let P be a quasi-compact smooth weak formal scheme over V. Let X be a closed subvariety of $P \otimes_V k$. Then (P, X, Id_X) is a widening for X/V. When there is no ambiguity we shall also use P to denote the widening (P, X, Id_X). This is less accurate but more economical.

Let $\{P_n\}$ be the universal enlargement associated to the widening P. See Definition 4.13. Let $\vec{P}$ be the topos defined in 4.9.

5.2. Definitions. (i) The relative site $(X/V)_{\mathrm{MW}}|_P$ is the site whose objects are arrows $T \to P$ of widenings, in which T is an enlargement of (X/V). The notion of coverings is defined in the obvious fashion. Let $(X/V)_{\mathrm{MW}}|_P^\sim$ be its sheaf topos. There is a natural morphism of topoi $j_P : (X/V)_{\mathrm{MW}}|_P^\sim \to (X/V)_{\mathrm{MW}}^\sim$.

(ii) We want to define a morphism of topoi φ_P from $(X/V)_{\mathrm{MW}}|_P^\sim$ to $P_{\mathrm{Zar}}^\sim$. Let $\mathcal{F}$ be a Zariski sheaf on P, $\mathcal{G}$ a sheaf on $(X/V)_{\mathrm{MW}}|_P$. Let U be an open weak formal subscheme of P. Then $(U, U \cap X, U \cap X \to X)$ is a widening which represents an object in the relative topos that is denoted by h_U. Let $\{U_n\}$ be its associated universal enlargement of U, so $h_U = \operatorname{colim} h_{U_n}$. Let $g : T \to P$ be a morphism of widenings, in which T is an enlargement. So g is an object in the relative site. Define

- $\varphi_P^{-1}(\mathcal{F})(g) = g^{-1}(\mathcal{F})(T)$, and
- $\varphi_{P,*}(\mathcal{G})(U) = \mathrm{Hom}(h_U, \mathcal{G}) = \lim_n \mathcal{G}(U_n)$.

If $\mathcal{F}$ is a coherent sheaf on P, define

$$\varphi_P^* \mathcal{F} = \varphi_P^{-1}\mathcal{F} \otimes_{\varphi_P^{-1}\mathcal{O}_P} j_P^{-1}\mathcal{O}^{\mathrm{an}}_{X/V}.$$

Clearly $\varphi_{P*}\varphi_P^*\mathcal{F} = \mathcal{F} \otimes_V K$.

(iii) Let $\mathcal{G} \in (X/V)_{\mathrm{MW}}|_P^\sim$. We define the functor, which is not a part of a morphism of topoi, φ_* to be the functor sending $\mathcal{G}$ to $(\mathcal{G}|_{P_n})_n$, that is, if U_n is an open of P_n, then $(U_n, U_n \cap (P_n \otimes_V k))$ is naturally an enlargement of (X/V) lying over P, and $\mathcal{G}|_{P_n}(U_n)$ is the value of $\mathcal{G}$ at $(U_n, (U_n \cap P_n \otimes_V k))$.

Lemma 5.3. *The functor φ_* is exact.*

Proof. This follows from the definition of covering in the Monsky–Washnitzer site. In fact, if we are given an exact sequence of abelian sheaves $\mathcal{F}^\bullet$, and a cocycle of the complex $\mathcal{F}^\bullet(U_n)$, then there exists an open covering of the enlargement U_n such that the restriction of this cocycle becomes coboundary we restricting to the smaller opens. Since covering are defined using Zariski open covering, this exactness implies the exactness of the restriction complex $\varphi_*(\mathcal{F}^\bullet)$. ◻

Since φ_* is not part of a morphism of topoi, we need the following lemma, whose proof is identical to [12, Lemma 0.4.1], that connects the cohomology of a sheaf $\mathcal{F}$ on $(X/V)_{\mathrm{MW}}|_P$ and the cohomology of $\varphi_*\mathcal{F}$ on the site $\vec{P}$.

Lemma 5.4. *The functor ϕ_* takes injective abelian sheaves to flasque abelian sheaves. Thereby for any abelian sheaf $\mathcal{F}$, we have* $\mathrm{H}^i((X/V)_{MW}|_P, \mathcal{F}) = \mathrm{H}^i(\vec{P}, \phi_*\mathcal{F})$.

5.5. Next we study the functor $j_{P,*}$ and its derived functors. In this paragraph we first discuss how to compute the direct image of j_P. Let $(T, i : Z \to T, z : Z \to X)$ be an absolutely affine enlargement. Let $\mathcal{G} = \varphi_P^* \mathrm{sp}_* \mathcal{F}$ for some coherent sheaf $\mathcal{F}$ on the weak formal scheme P. Then by definition $j_{P*}(\mathcal{G}) = \mathcal{G}(P \times T)$. Recall, $P \times T$ is the widening whose weak formal scheme is $P \times_V T$, which is again a weak formal scheme; the central part is $Z \times_{z,X} X = Z$. So we view Z as a closed subscheme of $P \times_V T$ via the product embedding $z \times i$.

However, since $P \times_V T$ is not an enlargement, we cannot "evaluate" a sheaf on it. What we could do is to take the Hom group

$$\mathrm{Hom}(\mathcal{G}, P \times T)$$

of sheaves, i.e., we regard $P \times_V T$ as a sheaf. On the other hand, we know $P \times_V T$ is the colimit of its universal enlargement, i.e., as sheaves we have

$$P \times T = \mathrm{colim}_n (P \times T)_n.$$

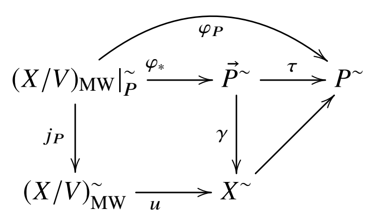
As $(P \times T)_n$ is an enlargement lying over P, we then conclude that

$$\mathrm{Hom}(\mathcal{G}, P \times T) = \lim \mathcal{G}(P \times T)_n = \lim \mathcal{F}(P \times T)_{n,K} = \mathcal{F}((Z)_{P \times_V T}).$$

From this we conclude that the direct image functor j_{P*} is computed by means of the *value of the sheaf on the tubular neighborhood*.

Lemma 5.6. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_P$-sheaf. Then $\mathcal{G} = \phi_P^* \mathcal{F}$ is j_P-acyclic, and $j_{P*}(\mathcal{G})$ is $u_{X/V}$-acyclic. Cf.* [12*, p.* 0.4.3].

Proof. We have a commutative diagram



of morphisms of topoi (except φ_*). Let $\mathcal{F}$ be a coherent sheaf on P. Then using definition one checks that $\tau^*(\mathcal{F} \otimes K) = \varphi_* \mathcal{G}$. By "Theorem B", Lemma 4.11, $R^i \gamma_*(\tau^*(\mathcal{F} \otimes K)) = 0$ for $i > 0$. Therefore we have

$$\begin{aligned} Ru_* R j_{P*} \mathcal{G} &= R\gamma_* R\varphi_* \mathcal{G} \\ &= \gamma_* \varphi_* \mathcal{G} \\ &= u_* j_{P*} \mathcal{G}. \end{aligned}$$

The acyclicity of $j_P \mathcal{G}$ with respect to $u_{X/V}$ then follows immediately from the spectral sequence of composition of functors and the acyclicity of $\mathcal{G}$ with respect to j_P.

Let us now prove the vanishing of $R^m j_{P*} \mathcal{G}$ for $m > 0$. To this end, it suffices to prove that for any absolutely affine enlargement (T, Z, z), the Zariski sheaf $R^m j_{P*} \mathcal{G}_T$ is zero. Consider the projection

$$\rho = \mathrm{pr}_2|_{(Z)_{P \times T}} : (Z)_{P \times T} \to T_K.$$

Then the Zariski sheaf $R^m j_{P*} \mathcal{G}_T$ equals

$$R^m \rho_* \mathcal{F}_K.$$

This can be seen by chasing a diagram of product widenings. The above sheaf vanishes for $m > 0$ since the projection is locally isomorphic to $D(0; 1^-)^N \times T_K \to T_K$ as dagger spaces, and the open polydisk has no cohomology by applying "Theorem B", Lemma 4.11. □

5.7. Let Ω^i_P be $\varphi_P^* \Omega^{i,\dagger}_{P/V}$ (see 5.2), i.e., the ring-theoretic pullback of the weakly completed de Rham complex of the weak formal scheme P.

On the weak formal scheme P, the exterior differentials $d : \Omega^{i,\dagger}_{P/V} \otimes K \to \Omega^{i+1,\dagger}_{P/V} \otimes K$ are only K-linear, not $\mathcal{O}_P \otimes K$-linear. Therefore the sheaves Ω^i_P do not form a complex in an obvious fashion. Our first mission is to define natural exterior differentials $d : \Omega^i_P \to \Omega^{i+1}_P$, and get a natural complex $\Omega^\bullet_P$ on the Monsky–Washnitzer site.

To this end, we need to define the sheaf relative differential forms for a morphism of dagger spaces. This process occupies 5.8 — 5.14. In this course, although not strictly needed, we use the notion of bornological spaces to make the treatment conceptual, but in the end bornology does not appear in our result. Bambozzi's thesis [1], which contains a complete index for terminologies and definitions, is our reference for bornological spaces and bornological algebras.

Let A be an affinoid dagger K-algebra. In the sequel, we fix a presentation $A = W_m/I$ where $I \subset T_n(\rho_0)$.

Definition 5.8. A *dagger* A-module is a bornological A-module M such that there exists a strict epimorphism $A^n \to M$ for some finite cardinal n.

Here is an Artin-Rees type lemma for dagger modules asserting that any finitely generated A-module has a canonical bornological module that is compatible with submodules.

Lemma 5.9. *If A is a K-dagger algebra, then each finitely generated A-module admits a unique bornology making it into a dagger A-module. Moreover, with respect to this canonical bornology, the category of finitely generated A-modules is equivalent to the category of dagger A-modules.*

Proof. See Corollaries 6.15, 6.16 of F. Bambozzi's thesis [1]. □

5.10. Let B be a dagger A-algebra. This means that B is of the form $W_{m+n}/\{IW_{m+n} + J\}$. One particular example of B is $A\langle t_1, \dots, t_n\rangle^\dagger = W_{m+n}/IW_{m+n}$. Let us arrange such that I, J are all convergent on the polydisk of radius ρ_0. Thus we can present B as a filtered colimit $B = \operatorname{colim}(B_\rho)$ with injective transition maps. Here $B_\rho = \mathfrak{T}_{m+n}(\rho)/\{I\mathfrak{T}_{m+n}(\rho) + J\}$. B is complete as a bornological k-space, as it is a colimit of complete Banach spaces; but it may not be complete as a "multiplicatively convex bornological algebra" in the sense of Bambozzi (which is defined as the direct limit of Banach k-algebras with monomorphic transition maps), as it is could contain torsion elements.

Let $B \otimes_A B$ be the completed tensor product obtained as the colimit of the completed tensor product $B_\rho \otimes_{A_\rho} B_\rho$ with the colimit bornology. Then we see B, $B\widehat{\otimes}_A B$ are all dagger algebras over k (by writing down a presentation of B).

5.11. Construction of $\Omega^1_{B/A}$. Let M be a dagger B-module. A *bounded* A-linear derivation is a derivation $D : B \to M$ that is also bounded with respect to the bornology of B inherited from the presentation and the canonical bornology of M.

Let M be a bornological B-module. Consider the algebra $B * M$ whose ambient B-module is

$$B \oplus M$$

with multiplication $(b + m)(b' + m') = bb' + bm' + b'm$. Then $B \oplus M$ is a bornological B-module with the product bornology. The multiplication of $B * M$ is bounded, since the action of B on M is bounded. The group of bounded A-derivations is identified with the subset of bounded A-algebra homomorphism from B into $B * M$ that reduces to the identity modulo M.

The multiplication defines a bounded map $\mu : B \otimes_A B \to B$ of bornological A-algebras. The maps $B \to B \otimes_A B$ defined by $d_0(b) = b \otimes 1$ and $d_1(b) = 1 \otimes b$ are bounded. Define $I = \mathrm{Ker}(\mu)$. Let $B^{(i)} = (B \otimes_A B)/I^{i+1}$. Clearly the bounded homomorphisms d_0 and d_1 are liftings of the identity $B = B^{(0)}$. Thus $\mathrm{d} = d_1 - d_0$ is a derivation valued in the ideal I. Since I is finitely generated, d is necessarily bounded.

Given a bounded A-linear derivation $D : B \to M$, where M is a bornological B-algebra. We get two bounded maps $B \to B * M$ given by $\mathrm{Id}_B + 0$ and $\mathrm{Id}_B + D$. This gives a bounded homomorphism, by the universal product of the completed tensor product

$$B \widehat{\otimes}_A B \to B * M$$

Note that I is mapped to M since the composition

$$B \widehat{\otimes}_A B \to B * M \to B$$

is none other than the multiplication map. Since $M^2 = 0$, I^2 is sent to zero, and we obtain a homomorphism of B-modules $I/I^2 \to M$. This proves that $\mathrm{d} : B \to I/I^2$ is the universal bounded derivation. Since we do not use usual Kähler differentials for dagger algebras, we will denote I/I^2 by $\Omega^1_{B/A}$ without ambiguity.

Lemma 5.12. *Let A be a dagger k-algebra. Let $B = A\langle t_1, \ldots, t_n\rangle^\dagger$. Then $\Omega^1_{B/A} = \bigoplus_{i=1}^n B \cdot \mathrm{d}t_i$.*

Proof. For any $1 < \rho < \rho_0$, define $B_\rho = A_\rho\langle t/\rho\rangle$. The maps $d_{i,\rho} : B_\rho \to B_\rho^{(1)}$ are defined and when $\rho < \rho'$, we have $d_{i,\rho'}|_{B_\rho} = d_{i,\rho}$. This means that the A_ρ-modules $\Omega^1_{B_\rho/A_\rho}$, the module of completed Kähler differential for B_ρ/A_ρ form a bornology of $\Omega^1_{B/A}$. Clearly this agrees with the canonical bornology as $\Omega^1_{B/A}$ is finitely generated over B by definition. The assertion now follows from the standard result for completed differentials, as the $\mathrm{d}t_i$ form a basis all $\Omega^1_{B_\rho/A_\rho}$ □

The module completed differentials clearly sheafifies (as it is defined by universal property). For a dagger space X over $\mathrm{Sp}(A)$, we can thus define $\Omega^1_{X/A}$. One can then develop the theory of smooth and étale morphism of dagger spaces. But for our purpose we only need to know the following two simple facts.

Lemma 5.13. *Let A be an affinoid dagger K-algebra. Let $X = \bigcup_{0<\eta<1} \mathrm{Sp}(A\langle t_1/\eta, \ldots, t_n/\eta\rangle^\dagger)$. Then $\Omega^1_{X/A}$ is the free $\mathcal{O}_X$-module generated by $\mathrm{d}t_i$.*

Proof. The differential forms $\mathrm{d}t_i$ are a compatible system of bases of the sheaf of relative differentials on the admissible covering $\mathrm{Sp}(A\langle t/\eta\rangle^\dagger)$. □

Lemma 5.14. *Let A be an affinoid dagger algebra. Let X be a dagger space. Form the product $X_A = X \times \mathrm{Sp}(A)$. Then $\Omega^1_{X_A/A} = \mathrm{pr}_1^*\Omega^1_{A/K}$.*

Proof. We can assume $X = \mathrm{Sp}(B)$ is affinoid. Let M be a bornological $A\widehat{\otimes}_K B$-module. Then it could be viewed as a bornological B-module in a natural way. Any A-linear bounded derivation from the tensor product into M gives rise a bounded K-linear derivation from B into M. Thereby we get a bounded B-linear homomorphism $\Omega^1_{B/K} \to M$ that lifts to a bounded A-linear homomorphism $A \otimes_K \Omega^1_{B/A} \to M$, in a unique fashion. $\square$

5.15. Definition. Let A be an affinoid dagger algebra. Let $f : X \to \mathrm{Sp}(A)$ be a morphism of dagger spaces. Then the universal A-linear bounded derivation $\mathrm{d} : \mathcal{O}_X \to \Omega^1_{X/A}$ prolongs to a complex $\Omega^\bullet_{X/A}$, whose arrows d are A-linear, and $\Omega^i_{X/A} = \bigwedge^i \Omega^i_{X/A}$. This complex is called the *relative (bounded) de Rham complex*.

5.16. Construction. Let $(T, i : Z \to T, z : Z \to X)$ be an absolutely affine enlargement. Let $T_K = \mathrm{Sp}(A)$ be the generic fiber of T. Since T is an enlargement, we have $(Z)_T = T_K$. We form the product

$$(T \times_V P, Z \xrightarrow{i,z} T \times_V P, z : Z \to X)$$

of widenings. Then according to 5.5, $j_{P*}\Omega^i_P(T) = \mathrm{pr}_2^*\Omega^i_{P_K}((Z)_{T\times_V P})$ where

$$\mathrm{pr}_2 : T_K \times P_K \to P_K$$

is the projection to the second factor. Since $(Z)_{T\times_V P}$ is an open subspace of $T_K \times P_K$, we have, by Lemma 5.14, that

$$j_{P*}\Omega^i_P(T) = \Omega^i_{(Z)_{T\times_V P}/T_K}((Z)_{T\times_V P}).$$

We then can use the exterior differentials, Definition 5.15, to define

$$\mathrm{d} : j_{P*}\Omega^i_P(T) \to j_{P*}\Omega^{i+1}_P(T).$$

Thereby we get a complex whose entries are $j_{P*}\Omega^i_P$, with differentials defined above. We denote this complex by $\mathrm{dR}^\bullet_P$.

Lemma 5.17. *The natural map*

$$\mathcal{O}^{\mathrm{an}}_{X/V} \to \mathrm{dR}^\bullet_P$$

is a quasi-isomorphism of sheaves on the Monsky–Washnitzer topos.

Proof. By the weak fibration theorem, and induction on relative dimension, it suffices to prove that the relative de Rham cohomology of $\mathrm{Sp}(A) \times D(0; 1^-)$ is quasi-isomorphic to A. This is a two term complex

$$\lim_{0<\eta<1} \mathrm{colim}_{1<\rho<r}\ A_\rho\langle t/\eta\rangle^\dagger \xrightarrow{\partial/\partial t} \lim_{0<\eta<1} \mathrm{colim}_{1<\rho<r}\ A_\rho\langle t/\eta\rangle^\dagger.$$

Set $R_\eta = \mathrm{colim}_{1<\rho<r}\ A_\rho\langle t/\eta\rangle^\dagger$. Pick an element $f \in R_\eta$. Then there exists ρ, such that $f \in A_\rho\langle t/\eta\rangle$ (without dagger), standard integration gives rise an element in $g \in A_\rho\langle t/\eta'\rangle$, where η' is a number slightly bigger than η, such that $\partial g/\partial t = f$. This means that the morphism ∂/∂_t on pro-object $\{R_\eta\}$ is surjective. The kernel equals A since we can embed the map in the ring of formal power series $A[\![t]\!]$ and check the kernel there. $\square$

Lemma 5.18. *We have an isomorphism of complexes of Zariski sheaves*

$$u_{X/V,*}(\mathrm{dR}^\bullet_P) \cong \mathrm{sp}_*\Omega^\bullet_{(Z)_P}.$$

Proof. Since $u_{X/V,*}$ and j_{P*} preserves global sections, we have

$$\Gamma(U, u_{X/V,*}j_{P*}\Omega^i_P) = \Omega^i_{(Z)_P}((Z)_P) = \Gamma(U, \mathrm{sp}_*\Omega^i_{(Z)_P})$$

for any Zariski open subset of X. Thus, the entries of the two complexes are equal. It remains to identify differentials. Let (T, Z, z) be an enlargement. The morphism $T \times P \to P$ is visualized by the morphism Consider the morphism

$$\begin{array}{ccc} (Z)_{P\times T} & \longrightarrow & (X)_P \\ \downarrow & & \downarrow \\ P_K \times T_K & \longrightarrow & P_K \end{array}$$

of dagger spaces. By Lemma 5.14, the global sections of the relative de Rham complex on $(Z)_{P\times T}$ and the global section of the absolute de Rham complex on $(X)_P$ has a natural chain map, and is functorial in T. Thus we get a natural morphism of chain complexes

$$\Gamma(X, \mathrm{sp}_*\Omega^\bullet_{(X)_P}) \to \Gamma(X, u_*\mathrm{dR}_P).$$

which is isomorphism on entries. The lemma follows since we can replace X by its Zariski opens. ◻

Corollary 5.19. *Let X be a k-variety. Let P be an admissible weak formal scheme over V whose completion is smooth. Assume that $P \otimes_V k$ contains X as a closed subscheme. Then the analytic cohomology of X with respect to P is isomorphic to the Monsky–Washnitzer cohomology of X.*

Proof. By Lemma 5.6, $u_{X/V,*}$ is acyclic with respect to sheaves $j_{P*}\Omega^i_P$, thereby

$$Ru_{X/V,*}\mathrm{dR}_P = u_{X/V,*}\mathrm{dR}_P.$$

By the Poincaré lemma 5.17, the Zariski cohomology of the left hand side equals the Monsky–Washnitzer cohomology; by Lemma 4.11 and Lemma 5.18 the Zariski cohomology of the right hand side equals the analytic cohomology. ◻

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