NEARBY CYCLES DO NOT GLUE

References.

- F. El Zein, D. T. Lê, and L. Migliorini. *A topological construction of the weight filtration* (2010). Example 2.0.5.
- M. Kashiwara and P. Schapira. *Sheaves on manifolds* (1990).
- J.-L. Verdier. *Spécialisation de faisceaux et monodromie modérée* (1983).

0.1. The problem. Let X be a complex manifold. Let D be a nonsingular Cartier divisor on X defined by a holomorphic function $f\colon \mathrm{X} \to \mathbf{C}$. Then we have the nearby cycle functor

$$\mathbf{R}\Psi_f\colon \mathrm{D}^{\mathrm{b}}(\mathbf{C}_{\mathrm{X}-\mathrm{D}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{C}_{\mathrm{D}}).$$

It turns out that *this functor depends on the choice of the defining function f of* D. For a different defining function g, it could happen that

$$\mathbf{R}\Psi_g(\mathrm{K}) \neq \mathbf{R}\Psi_f(\mathrm{K}) \text{ in } \mathrm{D}^{\mathrm{b}}(\mathbf{C}_{\mathrm{D}}), \quad \text{for some } \mathrm{K} \in \mathrm{D}^{\mathrm{b}}(\mathbf{C}_{\mathrm{X}-\mathrm{D}}).$$

Example (El Zein–Lê–Migliorini)**.** Consider $\mathrm{X} = \mathbf{C}_z \times \mathbf{C}^*_w$, $f(z,w) = z$, $g(z,w) = zw^{-1}$, with $\mathrm{D} = \{z = 0\}$. Let L be any local system on $\mathbf{C}^*_z$. Let $\mathrm{K} = f^*\mathrm{L}$. Then $\mathbf{R}\Psi_f(\mathrm{K})$ is a trivial local system on $\mathrm{D} \simeq \mathbf{C}^*$, while $\mathbf{R}\Psi_g(\mathrm{K})$ is not a trivial local system.

0.2. Verdier's functor. The example is best understood from the point of view of Verdier's *specialization functor.* Let us briefly recall the definition. Let X be a complex manifold, and Z a closed submanifold. Let $c\colon \widetilde{\mathrm{X}} \to \mathbf{A}^{1,\mathrm{an}}$ be the deformation to the normal bundle of Z. Thus $c^{-1}(\{0\})$ is a copy of the normal bundle $\mathrm{T_Z X}$ of Z, and $c^{-1}(\{t\}) \simeq \mathrm{X}$ whenever $t \neq 0$. Let $p\colon \widetilde{\mathrm{X}} - c^{-1}\{0\} \simeq \mathrm{X} \times \mathbf{C}^* \to \mathrm{X}$ be the natural projection.

Then the *specialization* of a complex $\mathrm{K} \in \mathrm{D}^{\mathrm{b}}(\mathbf{C}_{\mathrm{X}})$ is defined via a nearby cycle functor:

$$\nu_{\mathrm{Z}}(\mathrm{K}) = \mathbf{R}\Psi_c(p^{-1}\mathrm{K}) \in \mathrm{D}^{\mathrm{b}}(\mathbf{C}_{\mathrm{T_Z X}}).$$

Assume now Z is a smooth principal *divisor* defined by a function f. Then the differential $\mathrm{d}f$ defines a fiberwise linear function f' on the normal bundle. Since the fibers are one-dimensional, and since Z is nonsingular, f' is an isomorphism on each fiber. In particular, the locus $f' = 1$ defines a section of Z in $\mathrm{T_Z X}$:

$$s_f\colon \mathrm{Z} \to \mathrm{T_Z X}, \quad x \mapsto (x,\xi) \quad \text{such that } f'(x,\xi) = 1.$$

Now we can give Verdier's characterization of the nearby cycle functor.

Theorem (Verdier)**.** *In the situation above,* $\mathbf{R}\Psi_f(\mathrm{K}) = s_f^{-1}\nu_{\mathrm{Z}}(\mathrm{K})$.

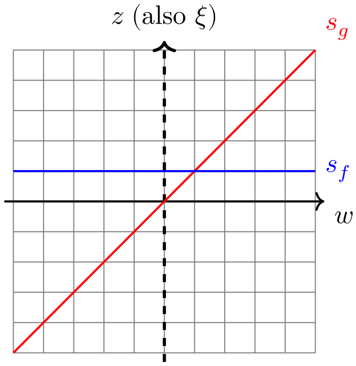
See the property SP6 mentioned in §§8, 9 of Verdier's article. We remark that the superscript P in Verdier's article stands for "moderate nearby cycle", which can be ignored in the present topological context. Verdier's theory is the complex analytic/ℓ-adic analogue of the microlocal theory of Kashiwara–Schapira.

0.3. Back to the example. We can now prove the assertions made in the example of §0.1. First, this is a linear situation, which means that the normal bundle $\mathrm{T_D X}$ is isomorphic to X: we can view the point (z, w) of X as a normal vector of $w \in \mathrm{D}$. It follows that the specialization $\nu_{\mathrm{D}}(\mathrm{K})$ is still the complex K.

The function f induces a fiberwise linear map $f' \colon \mathrm{T_D X} \to \mathbf{C}$ which sends (ξ, w) to ξ; while g induces $g' \colon (\xi, w) \mapsto \xi/w$. These two fiberwise linear functions induce sections

$$s_f \colon \mathrm{D} \to \{f' = 1\} \quad \text{and } s_g \colon \mathrm{D} \to \{g' = 1\}$$

of the normal bundle projection $\pi \colon \mathrm{T_D X} \to \mathrm{D}$, as shown in the picture below.



By Verdier's characterization of the nearby cycle functor,

$$\mathbf{R}\Psi_f(\mathrm{K}) = s_f^{-1}\mathrm{K}, \quad \mathbf{R}\Psi_g(\mathrm{K}) = s_g^{-1}\mathrm{K}.$$

In the diagram, the local system K is trivial in the horizontal direction (where z is constant), thus $\mathbf{R}\Psi_f(\mathrm{K})$ is trivial. But K is nontrivial in the vertical directions (hence nontrivial in the non-horizontal directions). The restriction of K to $s_g(\mathrm{D})$ is thus isomorphic to L, which is nontrivial.