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Elliptic surfaces with nonreduced Picard schemes

By Dingxin Zhang

Abstract

In these notes we give examples of an elliptic surface over a field of positive characteristic whose Picard scheme is nonreduced. The construction is essentially due to Katsura and Ueno [3, Exmaple 4.7] The proof that a suitable cover of the Katsura-Ueno surface has nonreduced Picard scheme relies on a result of Liedtke [5].

1. Elliptic surfaces. Let k be an algebraically closed field. An *elliptic surface* over k is a smooth, proper surface X together with a surjective morphism $f : X \to C$ to a smooth, proper curve, such that the geometric generic fiber is a smooth, connected curve with arithmetic genus 1.

Lemma 1.1. *Let $f : X \to C$ be an elliptic surface over an algebraically closed field k. Then*

(1) *the natural morphism $\mathscr{O}_C \to f_*\mathscr{O}_X$ is an isomorphism;*
(2) *all the geometric fibers of f are connected.*

Proof. This follows from the "connectedness theorem" (also known as the *Stein factorization*) as stated in [1, Théorème 4.3.1]. Let $f' : X \to Y$ be such that $f'_*\mathscr{O}_X = \mathscr{O}_Y$, and $g : Y \to C$ be the finite part of the Stein factorization. Since f' is proper and dominant, X is irreducible, it follows that Y is also irreducible. Since the geometric generic fiber of f is smooth and connected, and since the formation of Stein factorization is stable under arbitrary base change, we conclude that g is a finite separable morphism that is an isomorphism on the generic fibers. Since C is normal, g is finite and birational, it follows that g is an isomorphism. This proves (1). The assertion (2) follows from (1) and [1, Corollaire 4.3.2] □

REMARK 1.2 (Phenomenon of multiple fibers). Let $f : X \to C$ be an elliptic surface over an algebraically closed field k. Since the formation of coherent direct image is *not* compatible with arbitrary base change in general, it can happen that there exists an irreducible fiber F of f that is everywhere nonreduced, i.e., as a Weil divisor on X, F can be written as mD, where D is irreducible and is the reduced scheme of F.

Let the characteristic of k not be 2. Let E be an elliptic curve with hyperelliptic involution τ. Consider the product $E \times E$. It admits a diagonal action of $\mathbb{Z}/2$. The projection $\mathrm{pr}_1 : E \times E \to E$ is $(\mathbb{Z}/2)$-equivariant, hence descends to a morphism $\pi : X \to \mathbb{P}^1$ of the quotient varieties. Since the dimension is 1, X remains smooth. The morphism π is an "isotrivial family" away from the 4 points where τ is ramified, but the fibers over the ramification points of τ are multiple with multiplicity 2 whose reduced schemes are isomorphic to $\mathbb{P}^1$.

There is a standard method to deal with multiple fibers by means of normalized base change when the multiplicity is coprime to the characteristic of the ground field. Suppose $f : X \to C$ is an elliptic surface and mD is a multiple fiber of f over a point $s \in C$. Suppose $(m, \mathrm{char}(k)) = 1$, then we can resolve the multiple fiber by means of cyclic covering. Let $C' \to C$ be a finite morphism of smooth proper curves that formally locally looks like $z \mapsto z^m$ over $s \in C$. Let s' be fiber of $C' \to C$ over s. Then the normalization X' of the fiber product $X \times_C C'$ admits a morphism $f' : X' \to C'$ realizing X' an elliptic surface, such that the fiber over s' is no longer multiple. This method fails if we drop the hypothesis $(m, \mathrm{char}(k)) = 1$.

The examples we are about to construct and elliptic surfaces with some special sort of multiple fibers called wild fibers. It turns out that the when number of wild fibers increases, the Picard schemes become nonreduced.

2. Wild fibers. Let $f : X \to C$ be a surjective morphism from a surface onto a curve such that the canonical map $\mathcal{O}_C \to f_*\mathcal{O}_X$ is an isomorphism. Then by the theory of cohomology and base change, the formation of $R^1 f_* \mathcal{O}_X$ is compatible with arbitrary base change, and the Euler characteristics of $\mathcal{O}_{X_s}$ is constant for all points $s \in C$. It follows that there are finitely many points $s \in C$ over which

$$\mathrm{H}^1(X_s, \mathcal{O}_{X_s}), \quad \text{hence} \quad \mathrm{H}^0(X_s, \mathcal{O}_{X_s})$$

can jump up. Writing $R^1 f_* \mathcal{O}_X = \mathcal{L} \oplus \mathcal{T}$, where $\mathcal{L}$ torsion free and $\mathcal{T}$ torsion, we see the jumping phenomena only occur at the support of the torsion sheaf $\mathcal{T}$. When f is an elliptic fibration, $\mathcal{L}$ is an invertible sheaf, and all fiber fibers over $s \in \mathrm{Supp}(\mathcal{T})$ are multiple. We call these fibers *wild fibers* for the elliptic fibration f. According to Lemma 2.2 below, wild fibers are precisely the mulitple fibers $F = mD$ such that m is strictly larger than the order ν of the point $\mathcal{O}_X(D)$ in the Jacobian $\mathrm{Pic}^0(D)$. In §4 we will provide an example of wild fibers.

LEMMA 2.1. *Let Z be a smooth curve over an algebraically closed field k. Let $Z \subset X$ be an infinitesimal thickening of order 1 defined by an ideal $\mathcal{I}$. Suppose that $\mathcal{I} \cong \mathcal{O}_Z$ as a quasi-coherent sheaf of $\mathcal{O}_Z$-modules, then the conormal sequence*

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

splits. □

LEMMA 2.2. *Let C be a smooth, proper curve over an algebraically closed field k. Suppose we have an embedding $C \subset S$ of C into a smooth, proper surface over k, such that $C^2 = 0$ on S.*

(1) *If $\mathcal{O}_C(C) = \mathcal{O}_S(C)|_C$ is not torsion in $\mathrm{Pic}^0(C)$, then for all $m \in \mathbb{Z}$, $\mathrm{H}^0(C, \mathcal{O}_{mC})$ is a 1-dimensional vector space over k.*
(2) *If $\mathcal{O}_C(C)$ is torsion of order n, then $\mathrm{H}^0(C, \mathcal{O}_{mC})$ is a 1-dimensional vector space over k if and only if $m \leq n$.*

Proof. One uses induction and the exact sequence associated to the exact sequence

$$0 \to \mathcal{O}_C(-mC) \to \mathcal{O}_{mC} \to \mathcal{O}_{(m-1)C} \to 0$$

on the Zariski site of the scheme C, recalling that $\mathcal{O}_{mC} = \mathcal{O}_S/\mathcal{I}_C^{m+1}$. Item (1) follows immediately by induction, Item (2) follows from the splitting provided by Lemma 2.1 above. □

REMARK 2.3 (No wild fibers in characteristic zero). Retain the notations used above. Suppose the ground field k has characteristic zero. Then f does not admit any wild fiber. Indeed, by Kollár's theorem [4, Theorem 2.1(i)], the direct image sheaves $R^i f_* \omega_X$ are torsion free. Now we shall use this fact to prove our claim. By Grothendieck duality, we have

$$Rf_*(\omega_X[2]) = R\mathcal{H}om_{\mathcal{O}_C}(Rf_*\mathcal{O}_X, \omega_C[1]). \tag{2.4}$$

On the other hand, the theory of cohomology and base change tells us, locally on C, there is a two-term complex of free sheaves

$$\mathcal{G}_0 \xrightarrow{\varphi} \mathcal{G}_1$$

such that its cohomology sheaves computes the higher direct images of $\mathcal{O}_X$. Therefore $\mathrm{Ker}(\varphi) = \mathcal{O}_C$ and $\mathrm{Coker}(\varphi) = \mathcal{L} \oplus \mathcal{T}$. It follows that the complex

$$R\mathcal{H}om_{\mathcal{O}_C}(Rf_*\mathcal{O}_X, \omega_C[1])$$

is identified with

$$\mathscr{G}_0^\vee \otimes \omega_C \overset{\varphi^\dagger}{\longleftarrow} \mathscr{G}_1^\vee \otimes \omega_C$$

where the two terms are placed at cohomological degree -1 and -2. By (2.4), we thus have

$$f_*\omega_X = \mathrm{Ker}(\varphi^\dagger) = \mathscr{L}^\vee \otimes \omega_C, \text{ and } R^1 f_*\omega_X = \mathrm{Coker}(\varphi^\dagger) = \omega_C \oplus \mathscr{T}^*,$$

where $\mathscr{T}^*$ is the sheaf with same support as $\mathscr{T}$ but whose stalks are dual vector spaces of $\mathscr{T}$. By Kollár's theorem mentioned above, therefore, we infer that $\mathscr{T}^*$, hence $\mathscr{T}$, must be zero. Hence f does not admit wild fibers. The phenomenon of wild fibers also shows Kollár's torsion free theorem does not hold in positive characteristics.

3. Singularity of the Picard scheme. The nonreducedness of Picard scheme is reflected by the difference of the dimension of the reduction abelian variety and the dimension of the cohomology group $\mathrm{H}^1(X, \mathscr{O}_X)$, the Zariski tangent space of $\mathrm{Pic}^0(X)$ at the identity. For a scheme S, let

$$b_1(S) = \dim \mathrm{H}^1_{\text{ét}}(S, \mathbb{Q}_\ell),$$

then the Kummer theory implies that $b_1(X) = 2\dim \mathrm{Pic}^0(X)_{\mathrm{red}}$. Define the quantity h by $h = \dim \mathrm{H}^1(X, \mathscr{O}_X)$, then the difference $h - b_1(X)/2$ is positive if and only if $\mathrm{Pic}^0(X)$ is nonreduced. It turns out that the presence of wild fibers in an elliptic fibration is intimately related to the nonreducedness of the Picard scheme. The following result is due to Liedtke. See [5, Proposition 2.1].

LEMMA 3.1. *Let $f : X \to C$ be an elliptic surface. Suppose f has at least two wild fibers, then the Picard scheme of X is nonreduced.*

Proof. Write $R^1 f_*\mathscr{O}_X = \mathscr{L} \oplus \mathscr{T}$ as in §2. Then by the discussion in §2 we know $\mathrm{length}(\mathscr{T})$ equals the number w of wild fibers of f. We can compute the cohomology group $\mathrm{H}^1(X, \mathscr{O}_X)$ by means of Leray spectral sequence, which degenerates at E_2 by dimension reason. We thus have an exact sequence

$$0 \to \mathrm{H}^1(C, \mathscr{O}_C) \to \mathrm{H}^1(X, \mathscr{O}_X) \to \mathrm{H}^0(C, \mathscr{L}) \oplus \mathrm{H}^0(C, \mathscr{T}) \to 0.$$

It follows from the smoothness of the Jacobian that

$$h \geq \dim J(C) + w.$$

On the one hand, thanks to the a result of Katsura and Ueno, Lemma 3.2 below, we know $\dim J(C)$ is either the same as $\dim \mathrm{Alb}(X)$ or has 1 lower dimension. On the other hand, $\mathrm{Alb}(X)$ is the dual abelian variety to the reduction of $\mathrm{Pic}^0(X)$, hence they have the same dimension. When $w \geq 2$, $\dim J(C) + 2 > \dim \mathrm{Pic}^0(X)$, hence we have $h > b_1(S)/2$, and $\mathrm{Pic}^0(X)$ is nonreduced. □

LEMMA 3.2. *Let $f : X \to C$ be an elliptic surface. Let $\alpha : S \to \mathrm{Alb}(S)$ be the albanese morphism of S. Let $\varphi : C \to J(C)$ be the abel morphism of the curve C. Then the following conditions are equivalent:*

(1) *There is a fiber of f that is contracted by α.*
(2) $\mathrm{Alb}(S)$ *is isomorphic to* $J(C)$.

When this is not the case, we have $\dim \mathrm{Alb}(S) = \dim J(C) + 1$.

Proof. See [3, Lemma 3.4]. □

4. Producing wild fibers. We first provide an example of an elliptic surface with one wild fiber over a point $s \in \mathbb{P}^1$. Then by performing a covering unramified at s we can get an example of an elliptic surface S with more than one wild fibers (see Lemma 4.4). By Lemma 3.1, this implies that that S has nonreduced Picard scheme.

EXAMPLE 4.1. Let E be an *ordinary* elliptic curve over k. This means that E is an elliptic curve such that the absolute Frobenius pullback

$$F^* : \mathrm{H}^1(E, \mathscr{O}_E) \to \mathrm{H}^1(E, \mathscr{O}_E)$$

is zero (recall that this is a p-semilinear endomorphism). Equivalently, this means that the p-torsion of the abstract group $E(k)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Hence, we have an action of the reduced group scheme $\mathbb{Z}/p\mathbb{Z}$ on E.

The reduced group $\mathbb{Z}/p\mathbb{Z}$ also acts on the projective space $\mathbb{A}^1$ through $t \mapsto t+1$ by the virtue that $p = 0$ in k (this is known as the Artin-Schreier action). This action extends to an action on $\mathbb{P}^1$. The quotient scheme of $\mathbb{P}^1$ by this $\mathbb{Z}/p\mathbb{Z}$ action is still isomorphic to $\mathbb{P}^1$, and the quotient morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is étale away from the point at infinity.

Now consider the *diagonal action* of $\mathbb{Z}/p\mathbb{Z}$ on the product $\mathbb{P}^1 \times E$ defined by $\xi * (a, b) = (\xi * a, \xi * b)$. One checks that the diagonal action is a free action, hence the quotient morphism $\mathbb{P}^1 \times E \to X$ is étale. In particular, the quotient scheme X is smooth, projective, and has Kodaira dimension $-\infty$. With the diagonal action the projection to the first factor

$$\mathrm{pr}_1 : \mathbb{P}^1 \times E \to \mathbb{P}^1$$

is $\mathbb{Z}/p\mathbb{Z}$-equivariant. Let X be the quotient of $\mathbb{P}^1 \times E$ by the diagonal action, then the projection above induces a morphism

$$h : X \to \mathbb{P}^1.$$

For all $t \in \mathbb{P}^1 \setminus \{\infty\}$, the scheme-theoretic fiber of h over t is the scheme E, the fiber of h over $\{\infty\}$ is a multiple of multiplicity p, whose reduced structure is isomorphic to an elliptic curve E_∞, i.e., the quotient scheme of E by the reduced group scheme $E(k)[p] = \mathbb{Z}/p\mathbb{Z}$.

Lemma 4.2. *Let notations be as in Example 4.1. Then the multiple fiber $h^{-1}(\infty)$ is a wild fiber in the sense of §2.*

Proof. By the comments made in §2, the fiber $h^{-1}(\infty)$ is wild if and only if the normal sheaf $\mathcal{N}_{E_\infty/X} = \mathcal{O}_X(E_\infty)|_{E_\infty}$ of E_∞ in X is a trivial invertible sheaf. To see the triviality, we write down the normal sequence

$$0 \to T_{E_\infty} \to T_X|_{E_\infty} \to \mathcal{N}_{E_\infty/X} \to 0. \tag{4.3}$$

Since the quotient morphism $q : \mathbb{P}^1 \times E \to X$ is étale, we know it is flat, and there are natural identifications

$$q^* T_{E_\infty} \approx T_E, \quad q^* T_X \approx T_{\mathbb{P}^1 \times E}.$$

It follows that the pullback of (4.3) is precisely the normal sequence of E in $\mathbb{P}^1 \times E$. Hence $q^*\mathcal{N}_{E_\infty/X}$ is trivial. We claim that this implies the triviality of $\mathcal{N}_{E_\infty/X}$. Indeed, the triviality of $q^*\mathcal{N}_{E_\infty/X}$ implies that

$$[\mathcal{N}_{E_\infty/X}] \in \widehat{E}_\infty = \mathrm{Pic}^0(E_\infty)$$

is a k-valued point of the kernel of the dual isogeny of q. Since $\mathrm{Ker}(q) = \mathbb{Z}/p\mathbb{Z}$, the kernel of the dual isogeny is naturally isomorphic to the Cartier dual μ_p of $\mathbb{Z}/p\mathbb{Z}$ (See, for example, [6, §15, Theorem 1]). The group scheme μ_p being *local*, the only k-valued point of it is the identity. We win. □

Lemma 4.4. *Let $f : X \to C$ be an elliptic surface. Let $g : C' \to C$ be a finite, flat morphism that is unramified over the singular locus of f. If $f^{-1}(t)$ is a wild fiber of X, then for any $t' \in g^{-1}(t)$, the $X'_{t'}$ is a wild fiber of the elliptic surface $X' = X \times_C C'$.*

Proof. This follows from the general theory of the cohomology and base change. Since the related results are separated in different places in EGA III, we shall follow the result as exposed in [2, Chapter III]. Indeed, as $R^2 f_* \mathcal{O}_X$ vanishes, by [2, Theorem III.12.11(b)], the base change morphism

$$\varphi_t^1 : (R^1 f_* \mathcal{O}_X) \otimes \kappa(t) \to \mathrm{H}^1(X_t, \mathcal{O}_{X_t})$$

is an isomorphism for all points $t \in C$. Hence the formation of cohomology commutes with arbitrary base change. It follows that the torsion piece $\mathcal{T}'$ for $R^1 f'_* \mathcal{O}_{X'}$ is identified with the pullback $g^*\mathcal{T}$. The lemma now follows. □

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