SPECICALIZATION OF $\mathcal{D}$-MODULES

These notes discuss the notion of specialization of $\mathcal{D}$-modules. Section 1 treats some foundational matters concerning the canonical filtration (again, we reserve the name "Kashiwara–Malgrange filtration" for something more refined). The existence of these filtrations on holonomic modules will be given in Section 2.

References.

1) Björk, *Analytic $\mathcal{D}$-modules*.
2) Philippe Maisonobe and Zoghman Mebkhout, *Le théorème de comparaison pour les cycles évanescents*.

1. Fuchsian filtrations

Throughout this section, we fix a complex manifold X, be a closed submanifold Z of X. Denote by $\varpi : \mathrm{T_Z X} \to \mathrm{Z}$ the projection map of the normal bundle .

1.1. Fuchsian filtration of differential operators. We begin by putting a filtration on the ring of differential operators.

Definition 1. Define the *Fuchsian filtration* (with respect to the submanifold Z) on $\mathcal{D}_\mathrm{X}$ by

$$\mathrm{V}^i_\mathrm{Z}\mathcal{D}_\mathrm{X} = \mathrm{V}^i\mathcal{D}_\mathrm{X} = \{\mathrm{P} \in \mathcal{D}_\mathrm{X} : \forall j, \mathrm{P}\mathcal{I}^j_\mathrm{Z} \subset \mathcal{I}^{j+i}_\mathrm{Z}\},$$

here $\mathcal{I}^j_\mathrm{Z} = \mathcal{O}_\mathrm{X}$ if $j \leq 0$.

Description of the Fuchsian filtration in local coordinates. Let $(\boldsymbol{x}, \boldsymbol{t})$ be a local coordinate system of X, where $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\boldsymbol{t} = (t_1, \ldots, t_c)$, and Z is defined by $\boldsymbol{t} = 0$. Then locally an operator falls in $\mathrm{V}^j\mathcal{D}_\mathrm{X}$ if and only if it has an expansion

$$\mathrm{P}(\boldsymbol{x}, \boldsymbol{t}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{t}}) = \sum_{\alpha\in\mathbb{N}^d} \sum_{\substack{\beta,\gamma\in\mathbb{N}^c \\ |\beta|-|\gamma|\geq j}} h_{\alpha,\beta,\gamma}(\boldsymbol{x}, \boldsymbol{t})\partial^\alpha_{\boldsymbol{x}} \boldsymbol{t}^\beta \partial^\gamma_{\boldsymbol{t}},$$

where there are only finitely many nonzero $h_{\alpha,\beta,\gamma}(\boldsymbol{x}, \boldsymbol{t})$. Here we are using the standard shorthand for multi-indices. For instance,

$$\partial^\beta_{\boldsymbol{t}} = \left(\frac{\partial}{\partial t_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial t_c}\right)^{\beta_c}, \quad |\beta| = \beta_1 + \cdots + \beta_c.$$

To describe the associated graded pieces of the Fuchsian filtration, we introduce the notion of a Fuchsian field.

Definition 2. A tangent vector field θ on X is called a *Fuchsian field* of Z, if it satisfies the following property: for any local section f of $\mathcal{I}_\mathrm{Z}$,

$$\theta(f) \equiv f \mod \mathcal{I}^2_\mathrm{Z}.$$

Fuchsian field exists locally on X. For example, under a local coordinate $(\boldsymbol{x}, \boldsymbol{t})$ of X, Z being cut out by $\boldsymbol{t} = 0$, a Fuchsian field of Z is given by $\theta = \sum t_i \frac{\partial}{\partial t_i}$.

While Fuchsian fields exist on X only locally, a computation using coordinates shows that they all correspond to the same vector field on the normal bundle. Indeed, the Fuchsian vector field is the infinitesimal action of the scaling action of $\mathbb{G}_m$ on T_ZX, intrinsic to the normal bundle. Let ξ_i be the principal symbol of t_i, viewed as a local section of the conormal bundle T^*_ZX. Then $\boldsymbol{\xi} = (\xi_1, \dots, \xi_c)$ form a coordinate system of the fibers of $T_ZX \to Z$. Then the Fuchsian field on T_ZX is given by $\sum \xi_i \frac{\partial}{\partial \xi_i}$.

Let

$$A = \mathbb{C}\left[\xi_1, \dots, \xi_c; \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_c}\right]$$

be the Weyl algebra in the $\boldsymbol{\xi}$-variables. The Fuchsian field (on the normal bundle) θ acts on A naturally, giving rise to a grading to A, such that $\deg \xi_i = 1$, $\deg \partial/\partial\xi_i = -1$. We have ($A(k)$ being the set of elements of degree k in A)

$$A(0) \cong \mathbb{C}\left[\xi_1 \frac{\partial}{\partial \xi_1}, \dots, \xi_c \frac{\partial}{\partial \xi_c}\right], \quad \mathrm{Gr}^k_V \mathcal{D}_X \cong \mathcal{D}_Z \otimes_{\mathbb{C}} A(k), \quad \text{and} \quad \mathrm{Gr}^\bullet_V \mathcal{D}_X \cong \mathcal{D}_Z \otimes_{\mathbb{C}} A.$$

Let $\mathcal{D}_{[T_ZX]}$ be the subring of $\mathcal{D}_{T_ZX}$ consisting of differential operators on T_ZX which are "algebraic" in the fiber direction. In other words, a section of $\mathcal{D}_{[T_ZX]}$ is a differential operator which is polynomial in $\boldsymbol{\xi}$. Let $\varpi : T_ZX \to Z$ be the bundle projection. Then we have a natural isomorphism of graded rings:

$$\varpi_* \mathcal{D}_{[T_ZX]} = \mathrm{Gr}^\bullet_V \mathcal{D}_X.$$

1.2. Fuchsian filtrations on $\mathcal{D}$-modules.

Definition. Let M be a $\mathcal{D}_X$-module. A filtration $\mathrm{Fil}^\bullet M$ is called a *Fuchsian filtration* if it satisfies the following conditions.

1) For any integer k, $\mathrm{Fil}^k M$ is a coherent $V^0\mathcal{D}_X$-module;
2) $\bigcup_{k\in\mathbb{Z}} \mathrm{Fil}^k M = M$;
3) for any integers k, ℓ, $V^\ell \mathcal{D} \cdot \mathrm{Fil}^k M \subset V^{\ell+k} M$;
4) there exists $k \gg 0$, such that for all $i \in \mathbb{N}$, we have

$$V^i \mathcal{D} \cdot \mathrm{Fil}^k M = \mathrm{Fil}^{k+i} M \quad \text{and} \quad V^{-i}\mathcal{D} \cdot \mathrm{Fil}^{-k} M = \mathrm{Fil}^{-k-i} M.$$

If in addition the filtration satisfies

(KM) locally on X, the action of a Fuchsian field θ on $\mathrm{Gr}^k_{\mathrm{Fil}} M$ has a minimal polynomial, and the real parts of the eigenvalues of θ belong to $[k, k+1[$,

then we say $\mathrm{Fil}^\bullet M$ is a *canonical filtration* of M.

Remarks. 1) Any coherent $\mathcal{D}_X$-module M locally has a Fuchsian filtration. Indeed, let $\mathcal{D}^{\oplus m} \to M \to 0$ be a surjecitve map. Then the quotient filtration of $V^\bullet \mathcal{D}^{\oplus m}$ on M is a Fuchsian filtration.

Conversely, if $\mathrm{Fil}^\bullet M$ is a Fuchsian filtration on M, then locally there exists a surjective map $\mathcal{D}^{\oplus m} \to M$ such that the quotient filtration inherited from $V^\bullet \mathcal{D}$ is a shift of U. $V^0\mathcal{D}$-generators of $\mathrm{Fil}^i M$.

2) The condition (KM), stands for "Kashiwara and Malgrange", is independent of the choice of the Fuchsian field on X. For if θ and θ' are two Fuchsian fields, $\theta - \theta' \in \mathcal{I}^2\Theta_X$ acts by zero on $\mathrm{Gr}^\bullet_{\mathrm{Fil}} M$. Here Θ_X is the tangent sheaf on X. Alternatively, this could be seen

by noting that $\mathrm{Gr}^{\bullet}_{\mathrm{Fil}}\mathrm{M}$ is a graded $\mathcal{D}_{[\mathrm{T_Z X}]}$-module, hence a $\mathrm{Gr}^0_{\mathrm{V}}\mathcal{D}_{\mathrm{X}}$-module; but then "the" Fuchsian field θ is a section of $\mathrm{Gr}^0_{\mathrm{V}}\mathcal{D}_{\mathrm{X}}$.

We shall show momentarily (after seeing some 1-dimensional examples) that canonical filtration, if exists, is unique. It is customary to use the notation $\mathrm{V}^{\bullet}\mathrm{M}$ to denote the canonical filtration of M.

Examples. Let $\mathrm{X} = \Delta$ be a small disk, and $\mathrm{Z} = \{0\}$ the origin.

1) Let M be an integrable connection on Δ^*, meromorphic at 0. Then the trivial filtration $\mathrm{Fil}^k\mathrm{M} \equiv \mathrm{M}$, while satisfies (2 – 4) and condition (KM), can violate the coherent condition (1). For instance, the trivial connection $\mathcal{O}[*0]$ is not coherent over $\mathrm{V}^0\mathcal{D}$, as $\mathrm{V}^0\mathcal{D}$ will never increase the pole order.

2) Let $\mathcal{D}\delta = \mathcal{D}/\mathcal{D}t$ be the $\mathcal{D}$-module supported at 0. Then the quotient filtration induced from the quotient map is the canonical filtration of $\mathcal{D}\delta$. Thus $\mathrm{V}^k\mathcal{D}\delta$ is nonzero only when k is negative; if $k \geq 1$, $\mathrm{V}^{-k}\mathcal{D}\delta$ is the $\mathbb{C}$-linear combination of $\delta, \delta^2, \ldots, \delta^k$, where δ^p is the image of ∂_t^p. It follows that $\mathrm{Gr}^{-k}_{\mathrm{V}}\mathcal{D}\delta = \mathbb{C}\cdot[\delta^k]$ is 1-dimensional.

3) Sometimes the trivial filtration *can* give the canonical filtration, due to the presence of irregular singularity. Let $\mathrm{M} = \mathcal{D}/\mathcal{D}(t^2\partial_t - 1)$. Then M is an integrable connection on Δ^* meromorphic but *irregular* at 0. Let us show M possesses a canonical filtration which is however trivial.

Consider the quotient filtration of M inherited from $\mathrm{V}^{\bullet}\mathcal{D}$. Let v be the image of $1 \in \mathcal{D}_{\mathrm{X}}$. Thus $v \in \mathrm{V}^0\mathrm{M}$, and satisfies $t^2\partial_t v = v$. Thus

$$v = (t^2\partial_t)^k v, \quad \text{hence} \quad v \in \bigcap_{k\in\mathbb{Z}} \mathrm{V}^k\mathrm{M}.$$

Since v generates M as an $\mathcal{O}[*0]$-module, we conclude that $\mathrm{M} = \bigcap_{k\in\mathbb{Z}}\mathrm{V}^k\mathrm{M}$, and the quotient filtration is trivial. Since the quotient filtration is a Fuchsian filtration by fiat, and satisfies the property (KM) vacuously, it is the canonical filtration.

4) Let M be a $\mathcal{D}$-module *regular* at 0 (this means that $\mathrm{M}[t^{-1}]$ is an integrable connection on Δ^*, which is meromorphic and regular at 0). Let us show the canonical filtration of M, if exists, is nontrivial.

The module M fits into a short exact sequence

$$0 \to (\mathcal{D}\delta)^{\oplus p} \to \mathrm{M} \to \mathrm{M}[t^{-1}] \to (\mathcal{D}\delta)^{\oplus q} \to 0.$$

By the strictness of the canonical filtration to be proven in §1.3, Lemma 3, it suffices to assume $\mathrm{M} = \mathrm{M}[t^{-1}]$ is a regular meromorphic connection. In this case, let $\mathrm{T_Z X^D}$ be the canonical lattice of M (discussed in a previous post). Then one checks directly that $\mathrm{V}^k\mathrm{M} = t^k\mathrm{T_Z X^D}$ is the canonical filtration on M. By the uniqueness of the canonical filtration to be proven below, we see that its canonical filtration cannot be trivial.

Lemma. *Let* M *be a* $\mathcal{D}_{\mathrm{X}}$*-module. Then* M *has at most one canonical filtration.*

Proof. The proof is virtually the same as the uniqueness of the canonical lattice we treated a while ago. Let $\mathrm{V}^{\bullet}\mathrm{M}$ and $\mathrm{U}^{\bullet}\mathrm{M}$ be two canonical filtrations of M. Fix k. Since $\mathrm{V}^k\mathrm{M}$ is coherent over $\mathrm{V}^0\mathcal{D}$, and since $\mathrm{M} = \bigcup_{\ell\in\mathbb{Z}}\mathrm{U}^{\ell}\mathrm{M}$, locally on X we can find ℓ such that a set S of $\mathrm{V}^0\mathcal{D}$-generators of $\mathrm{V}^k\mathrm{M}$ is contained in $\mathrm{U}^{\ell}\mathrm{M}$. Thence

$$\mathrm{V}^k\mathrm{M} = \mathrm{V}^0\mathcal{D}\cdot\mathrm{S} \subset \mathrm{V}^0\mathcal{D}\cdot\mathrm{U}^{\ell}\mathrm{M} \subset \mathrm{U}^{\ell}\mathrm{M}.$$

We can choose ℓ to be minimal, so that V^kM is not contained in $U^{\ell+1}M$. Thus the map $\iota\colon \mathrm{Gr}_V^k M \to \mathrm{Gr}_U^\ell M$ is nonzero.

Since the Fuchsian field θ acts on both associated graded, and since the image of the map ι is nonzero, θ has an eigenvector in $\mathrm{Im}\,\iota$, whose eigenvalue must have real part lying in $[k, k+1[\, \cap\, [\ell, \ell+1[$. This forces $k = \ell$, i.e., locally $V^kM \subset U^kM$. Since U and V are symmetric, we get the reverse inclusion as well. □

1.3. Specializable $\mathcal{D}$-modules.

Definition 1. Let M be a coherent $\mathcal{D}_X$-module. We say M is *specializable* along Z, if there exists, locally on X, a Fuchsian filtration $\mathrm{Fil}^\bullet M$ satisfying the following property:

(S) there exists, locally on X, a monic polynomial $b(s) \in \mathbb{C}[s]$, such that $b(\theta - k)$ acts trivially on $\mathrm{Gr}_{\mathrm{Fil}}^k M$.

Clearly, if M admits a canonical filtration, then $V^\bullet M$ satisfies (S) (even globally). Conversely, we shall show that any specializable module has canonical filtration.

Property (S) appears rather bizarre. Here is an explanation: we regard M as a higher dimensional analogue of an integrable connection over Δ^* meromorphic along 0, then Fil^0M is an analogue of a "saturated lattice" Λ of a meromorphic connection over Δ. In the 1-dimensional case, as $\Lambda/t\Lambda$ is finite dimensional, the residue of $t\partial_t$ trivially has a minimal polynomial $b(s)$; moreover the minimal polynomial of the residue of $t\partial_t$ with respect to the lattice t^kT_ZX is none other than $b(s-k)$.

As in the 1-dimensional case, one can perform shearing transformation to turn the filtration $\mathrm{Fil}^\bullet M$ into the canonical filtration.

Lemma 2 (Shearing transformations)**.** *Let* M *be a* $\mathcal{D}_X$*-module specializable along* Z*. Then* M *has a canonical filtration.*

Proof. Let $\Gamma^\bullet M$ be the filtration which satisfies the property (S). Assume that the polynomial b admits a factorization $b = b_1 \cdot b_2$. Define a new filtration $\Theta^kM = \Gamma^{k+1}M + b_1(\theta - k)\Gamma^kM$. Let $B(s) = b_1(s-1)b_2(s)$, then

$$\begin{aligned} B(\theta - k)\Theta^kM &= b_2(\theta-k)b_1(\theta-k-1)\Gamma^{k+1}M + b_1(\theta-k-1)b(\theta)\Gamma^kM \\ &\subset b_2(\theta-k)b_1(\theta-k-1)\Gamma^{k+1}M + b_1(\theta-k-1)\Gamma^{k+1}M \\ &\subset \Theta^{k+1}M. \end{aligned}$$

Thus $\Theta^\bullet M$ is another Fuchsian filtration of M, but this time the annihilating polynomial becomes $B(s) = b_1(s-1)b_2(s)$. Repeating this procedure yields a canonical filtration. □

Next we turn to the definition of the *specialization* of a specializable module. To this end we need the following lemma about the strictness of the canonical filtration.

Lemma 3. *Let* $\varphi\colon M \to N$ *be a morphism of specializable* $\mathcal{D}$*-modules, then* φ *is strictly compatible with the canonical filtrations.*

Proof. It is easy to see (from the snake lemma) that any quotient filtration on any quotient $\mathcal{D}$-module of M is a canonical filtration; similarly, if N' is a coherent $\mathcal{D}$-submodule of N, then the filtration $V^\bullet N \cap N'$ is a canonical filtration on N'. The lemma follows from these two simple observations. □

Let M be a coherent $\mathcal{D}_X$-module specializable along a closed submanifold Z of X. Let $T_Z X$ be the normal bundle $T_Z X$ of Z. Then the graded $\mathrm{Gr}_V^\bullet \mathcal{D}$-module $\mathrm{Gr}_V^\bullet M$ gives rise to a graded $\mathcal{D}_{[T_Z X]}$-module.

Definition 4. Define $\nu_Z(M) = \mathcal{D}_{T_Z X} \otimes_{\mathcal{D}_{[T_Z X]}} \mathrm{Gr}_V^\bullet M$. The $\mathcal{D}_{T_Z X}$-module $\nu_Z(M)$ is called the *specialization* of M along Z.

Lemma 3 ensures that the functor ν_Z from the category of Z-specializable $\mathcal{D}_X$-modules into the category $\mathcal{D}_{T_Z X}$-modules is an *exact* functor.

Note. The Fourier–Laplace transform $\mathcal{F}(\nu_Z(M))$ of $\nu_Z(M)$, which is a $\mathcal{D}$-module on the *conormal bundle* $T_Z^* X$, is called the *microlocalization* of M along Z. We denote the microlocalization of M along Z by $\mu_Z(M)$.

Example. Assume c is the codimension of Z in X. Let M be a coherent $\mathcal{D}_X$-module supported on Z. By Kashiwara's theorem, $M = \int_{Z\to X} N$ for some coherent $\mathcal{D}_Z$-module. We shall show M is specializable along Z.

Let $i\colon Z \to X$ be the embedding, and assume Z is locally defined by $\boldsymbol{t} = 0$ in some coordinate system. Then M is locally of the form $\bigoplus_{v\in\mathbb{Z}^c_{\geq 1}} \left(\frac{\partial}{\partial \boldsymbol{t}}\right)^v i_* N$. One checks that the canonical filtration is given by $V^{-k}M = \bigoplus_{|v|\leq k} \left(\frac{\partial}{\partial \boldsymbol{t}}\right)^v i_* N$. The $\mathcal{D}_Z$-module N can be recovered from M by taking the associated graded: $N \simeq \mathrm{Gr}_V^{-c} M$.

Let $\mathbf{0}\colon Z \to T_Z X$ be the zero section. Then one checks that $\nu_Z(M) \simeq \int_{\mathbf{0}} N$.

2. Specializations of holonomic modules

The purpose of this section is to prove that every holonomic $\mathcal{D}$-module is specializable along any closed submanifold.

Theorem. *Let* M *be a* holonomic $\mathcal{D}_X$*-module. Then:*

1) M *is specializable along* Z*, and*
2) $\nu_Z(M)$ *is a holonomic* $\mathcal{D}_{[T_Z X]}$*-module.*

2.1. Characterization of holonomic modules.

A $\mathcal{D}$-module M is said to be *holonomic* if its characteristic variety is Lagrangian, or equivalently that the irreducible components of its characteristic variety $\mathrm{Ch}(M)$ are n-dimensional, where $n = \dim X$.

Definition 1. The *codimension*, or *depth*, or, *grade*, $j(M)$ of a $\mathcal{D}$-module M is the smallest number j such that

$$\mathcal{E}xt^j_{\mathcal{D}}(M, \mathcal{D}) \neq 0.$$

There is a related concept, the *projective dimension* of M, denoted by $pd(M)$. It is the largest number d such that the functor $\mathcal{E}xt^d_{\mathcal{D}}(M, \cdot)$ is nonzero. Thus the right $\mathcal{D}$-module $\mathcal{E}xt^j_{\mathcal{D}}(M, \mathcal{D})$ are nonzero only in the range $[j(M), pd(M)]$

Let us explain how to get information of the $\mathcal{E}xt^j_{\mathcal{D}}(M, \mathcal{D})$ using *commutative algebra*. We equip $\mathcal{D}$ with its order filtration $F_\bullet\mathcal{D}$, and endow, locally on X, M with a good filtration $F_\bullet M$.

Lemma 2. *Locally on* X*, there is an exact sequence of filtered* $\mathcal{D}$*-modules*

$$\cdots \to F_\bullet P_1 \to F_\bullet P_0 \to F_\bullet M \to 0$$

such that the maps are strictly compatible with filtrations, and that each $\mathrm{F}_\bullet\mathrm{P}_i$ *is a finite direct sum of shifts of* $\mathrm{F}_\bullet\mathcal{D}$.

The proof is omitted. A filtration as in Lemma 2 is said to be a *filtered free resolution* of M.

Since a filtered free resolution is in particular a free resolution, we know that the complex of right $\mathcal{D}$-modules $\mathcal{H}om_{\mathcal{D}}(\mathrm{P}_\bullet, \mathcal{D})$ computes the $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$.

Each entry $\mathcal{H}om_{\mathcal{D}}(\mathrm{P}_i, \mathcal{D})$ is a free right $\mathcal{D}$-module, and the summands are endowed with shifts of the order filtration. It should be noted that the arrows of the complex $\mathcal{H}om_{\mathcal{D}}(\mathrm{P}_\bullet, \mathcal{D})$ are not necessarily strictly compatible with filtrations any more. Nevertheless, on each of the cohomology module $\mathcal{E}xt^i_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ we obtain a good filtration. In fact, one can show that this induced filtration only depends on the filtration $\mathrm{F}_\bullet\mathrm{M}$ on M, not on the particular filtered free resolution.

For each filtered free resolution $\mathrm{P}_\bullet \to \mathrm{M}$, the complex $\mathcal{H}om_{\mathcal{D}}(\mathrm{P}_\bullet, \mathcal{D})$ is a filtered complex:

$$\mathrm{Fil}_m\mathcal{H}om_{\mathcal{D}}(\mathrm{P}_k, \mathcal{D}) = \{f \in \mathcal{H}om_{\mathcal{D}}(\mathrm{P}_k, \mathcal{D}) : f(\mathrm{F}_n\mathrm{P}_k) \subset \mathrm{F}_{n+m}\mathcal{D}\}.$$

Hence there is a spectral sequence

$$\mathrm{E}_1^{-p,q} = \mathrm{H}^{-p+q}(\mathrm{Gr}^{\mathrm{Fil}}_p\mathcal{H}om(\mathrm{P}_\bullet, \mathcal{D})) \Rightarrow \mathcal{E}xt^{-p+q}_{\mathcal{D}}(\mathrm{M}, \mathcal{D}).$$

Since $\mathrm{Gr}^{\mathrm{Fil}}_p\mathcal{H}om(\mathrm{P}_\bullet, \mathcal{D})$ can be identified with the set of homomorphisms from $\mathrm{Gr}^{\mathrm{F}}\mathrm{P}$ into $\mathrm{Gr}^{\mathrm{F}}\mathcal{D}$ which are homogeneous of degree p, and since $\mathrm{Gr}^{\mathrm{F}}\mathrm{P} \to \mathrm{Gr}^{\mathrm{F}}\mathrm{M}$ is a free $\mathrm{Gr}^{\mathrm{F}}\mathcal{D}$-resolution, the spectral sequence above can be written as

$$\mathrm{E}_1^{-p,q} = \mathcal{E}xt^{-p+q}_{\mathrm{Gr}^{\mathrm{F}}_\bullet\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}_{\bullet-p}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}_\bullet\mathcal{D}).$$

Set

$$(*) \qquad \mathrm{E}_1^k = \mathrm{E}_1^k(\mathrm{M}) = \bigoplus_{-p+q=k} \mathcal{E}xt^k_{\mathrm{Gr}^{\mathrm{F}}_\bullet\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}_{\bullet-p}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}_\bullet\mathcal{D}) = \mathcal{E}xt^k_{\mathrm{Gr}^{\mathrm{F}}\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}\mathcal{D}).$$

This is a graded $\mathrm{Gr}^{\mathrm{F}}\mathcal{D}$-module, summing to the ungraded Ext-modules. The spectral sequence gives then a collection of graded $\mathrm{Gr}^{\mathrm{F}}\mathcal{D}$-modules E^k_r, and maps $d\colon \mathrm{E}^k_r \to \mathrm{E}^{k+1}_r$, such that E^k_{r+1} is the corresponding cohomology module. Since the spectral sequence converges, we see $\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ is a subquotient of $\mathcal{E}xt^k_{\mathrm{Gr}^{\mathrm{F}}\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}\mathcal{D})$.

Lemma 3. *Let* M *be a coherent* $\mathcal{D}$*-module.*

1) $\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D}) = 0$ *for all* $k < \operatorname{codim}\mathrm{Ch}(\mathrm{M})$.
2) $\operatorname{codim}\mathrm{Ch}(\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})) \geq k$.

Proof. The problem being local, we can choose a good filtration $\mathrm{F}_\bullet\mathrm{M}$ of M. It follows from commutative algebra that for $k \leq \operatorname{codim}\mathrm{Ch}(\mathrm{M})$ that

$$\mathcal{E}xt^k_{\mathrm{Gr}^{\mathrm{F}}\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}\mathcal{D}) = 0.$$

The lemma follows from the fact that $\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ is a subquotient of $\mathcal{E}xt^k_{\mathrm{Gr}^{\mathrm{F}}\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}\mathcal{D})$. This proves (1).

Still denote by $\mathrm{F}_\bullet$ the good filtration of $\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ inherited from that of M. From the spectral sequence $\mathrm{E}_1^k \Rightarrow \mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ we also learned that the associated graded $\mathrm{Gr}_{\mathrm{F}}\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ is a subquotient of $\mathrm{E}_1^k = \mathcal{E}xt^k_{\mathrm{Gr}^{\mathrm{F}}\mathcal{D}}(\mathrm{Gr}^{\mathrm{F}}\mathrm{M}, \mathrm{Gr}^{\mathrm{F}}\mathcal{D})$. In particular, the characteristic variety of the right $\mathcal{D}$-module $\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})$ is contained in the support of $\mathrm{E}_1^k(\mathrm{M})$. From commutative algebra, this support has codimension $\geq k$. It follows that $\operatorname{codim}\mathrm{Ch}(\mathcal{E}xt^k_{\mathcal{D}}(\mathrm{M}, \mathcal{D})) \geq k$. □

Using Lemma 3 we can study the *biduality spectral sequence* associated with a coherent $\mathcal{D}$-module M. Locally we choose a free resolution $P_\bullet \to M$, then the dual $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(M, \mathcal{D})$ is calculated by $\mathcal{H}om_{\mathcal{D}}(P_\bullet, M)$. Each right $\mathcal{D}$-module $\mathcal{H}om_{\mathcal{D}}(P_i, \mathcal{D})$ is again free, but the cohomology sheaves, i.e., $\mathcal{E}xt^i_{\mathcal{D}}(M, \mathcal{D})$, are by no means free. Thus one cannot take $\mathcal{H}om_{\mathcal{D}}(-, \mathcal{D})$ again to calculate the double dual. Choose a double complex $Q \to \mathcal{H}om_{\mathcal{D}}(P, \mathcal{D})$ which also resolves the $\mathcal{E}xt$-modules. Then we get a (fourth quadrant) double complex $\check{Q} = \mathcal{H}om_{\mathcal{D}}(Q, \mathcal{D})$. By definition, $\check{Q}$ is isomorphic to M in the derived category. But the construction gives a spectral sequence

$$E^{p,q}(M) := E_1^{p,-q} = \mathcal{E}xt^p_{\mathcal{D}}(\mathcal{E}xt^q_{\mathcal{D}}(M, \mathcal{D}), \mathcal{D}).$$

By Lemma 3, it follows that $E^{p,q} = 0$ if $q < p$.

This spectral sequence is referred to as the *biduality spectral sequence*. The spectral sequence gives a filtration B^iM on M, called the *codimension filtration* of M. such that $\mathrm{Gr}^p_B M$ is a subquotient of $E^{p,p}$.

Let $j = j(M)$ be the codimension of M. Then the first lowest possible nonzero entry in E_1 is $E^{j,j}$. The following lemma shows that it is indeed nonzero.

Lemma 4. *Let $j = j(M)$ be the codimension of* M. *Then we have* $E^{j,j} \neq 0$.

Proof. Assume that $E^{j,j} = 0$. By Lemma 3, for each $k > j$, the codimension of $\mathrm{Ch}(E^{k,k})$ is $> j$. It follows that the characteristic varieties of $\mathcal{D}$-modules $\mathrm{Gr}^k_B M$ have codimension $> j$. From this we deduce that the codimension of $\mathrm{Ch}(M)$ is $> j$. It follows from Lemma 3 again that $\mathcal{E}xt^j_{\mathcal{D}}(M, \mathcal{D}) = 0$, which contradicts with the fact that j is the codimension of M. □

Theorem. *Let* M *be a nonzero coherent $\mathcal{D}$-module. Then* M *is holonomic if and only if* $j(M) = \dim X$*, i.e.,* $\mathcal{E}xt^k_{\mathcal{D}}(M, \mathcal{D}) = 0$ *for all* $k \neq \dim X$.

Proof. Suppose first that M is holonomic. If $k < \dim X = \operatorname{codim} \mathrm{Ch}(M)$, then

$$\mathcal{E}xt^k_{\mathcal{D}}(M, \mathcal{D}) = 0$$

by Lemma 3(1). If $k > \dim X$, then by Lemma 3(2), the characteristic variety of the right $\mathcal{D}$-module $\mathcal{E}xt^k_{\mathcal{D}}(M, \mathcal{D})$ has codimension $> \dim X$. By Bernstein's inequality, we must have $\mathcal{E}xt^k_{\mathcal{D}}(M, \mathcal{D}) = 0$.

Conversely, suppose $\mathcal{E}xt^k_{\mathcal{D}}(M, \mathcal{D})$ is nonzero only for $k = \dim X$. Since the characteristic variety of the right $\mathcal{D}$-module $\mathcal{E}xt^{\dim X}_{\mathcal{D}}(M, \mathcal{D})$ is contained in E_1^n, and since $\operatorname{codim} E_1^n \geq n$, we conclude from Bernstein's inequality that $N = \mathcal{E}xt^{\dim X}_{\mathcal{D}}(M, \mathcal{D})$ is holonomic.

In particular, the "only if" part of the theorem, applying to N, implies that the left $\mathcal{D}$-modules $\mathcal{E}xt^k_{\mathcal{D}}(N, \mathcal{D})$ are zero except $k = \dim X$. From the biduality spectral sequence

$$\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{E}xt^{-j}_{\mathcal{D}}(M, \mathcal{D}), \mathcal{D}) \Rightarrow M$$

we conclude that $M = \mathcal{E}xt^{\dim X}_{\mathcal{D}}(N, \mathcal{D})$, which is holonomic by the argument of the previous paragraph. □

Definition 5. Let M be a coherent $\mathcal{D}$-module. Then the dual complex $\mathbb{D}M$ of M is the complex of left $\mathcal{D}$-modules associated with the complex of right $\mathcal{D}$-modules $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(M, \mathcal{D})[\dim X]$.

The theorem above (or rather its proof) implies that a coherent $\mathcal{D}$-module M is holonomic if and only if $\mathbb{D}M$ is a plain $\mathcal{D}$-module. In this case, $\mathbb{D}M$ is also necessarily holonomic. Moreover, $\mathbb{D}\mathbb{D}M = M$.

2.2. Specialization of holonomic modules. Let M be a coherent $\mathcal{D}$-module. Let $\mathrm{U}^\bullet\mathrm{M}$ be a Fuchsian filtration of M. Thus $\mathrm{Gr}_\mathrm{U}^\bullet\mathrm{M}$ is a graded $\mathrm{Gr}_\mathrm{V}^\bullet\mathcal{D}$-module.

Theorem. *Assume that* M *is holonomic. Then for any Fuchsian filtration* $\mathrm{U}^\bullet\mathrm{M}$ *of* M*, and any* $k \neq \dim \mathrm{X}$*,*

$$\mathcal{E}xt^k_{\mathrm{Gr}_\mathrm{V}\mathcal{D}}(\mathrm{Gr}_\mathrm{U}\mathrm{M}, \mathrm{Gr}_\mathrm{V}\mathcal{D}) = 0.$$

In particular, $\mathrm{Gr}_\mathrm{U}\mathrm{M}$ *is a holonomic* $\mathrm{Gr}_\mathrm{V}\mathcal{D}$*-module.*

Proof. The Fuchsian filtration gives rise to a spectral sequence (this requires some argument, as the filtration is not bounded)

$$\mathrm{E}_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}^p\mathcal{H}om_\mathcal{D}(\mathrm{P}_\bullet, \mathcal{D})),$$

where $\mathrm{P}_\bullet \to \mathrm{M}$ is a filtered resolution of $\mathrm{U}_\bullet\mathrm{M}$. Moreover, we have

$$\mathrm{E}_1^k(\mathrm{M}) = \bigoplus_{p+q=k} \mathrm{E}_1^{p,q} = \mathcal{E}xt^k_{\mathrm{Gr}_\mathrm{V}\mathcal{D}}(\mathrm{Gr}_\mathrm{U}\mathrm{M}, \mathrm{Gr}_\mathrm{V}\mathcal{D})$$

as in §2.1, and $\mathrm{E}_\infty^k = \mathrm{Gr}_\mathrm{U}\mathcal{E}xt^k_\mathcal{D}(\mathrm{M}, \mathcal{D})$.

Let $k = j(\mathrm{Gr}_\mathrm{U}\mathrm{M})$. Assume $k < n$. We shall derive a contradiction. Since M is holonomic, $\mathrm{E}_\mathrm{N}^k = 0$ for N sufficiently large. But there exist exact sequences

$$0 \to \mathrm{E}_{r+1}^k \to \mathrm{E}_r^k \to \mathrm{B}_r^{k+1} \to 0,$$

where the $(k+1)$-coboundaries B_r^k are subquotients of E_1^{k+1}. Since the codimension of E_1^{k+1} is at least $k+1$, it follows from a reverse induction that E_1^k has codimension $\geq k+1$. But this contradicts with §2.1, Lemma 4. □

2.3. Proof of the Theorem. To prove that M is specializable, let $\mathrm{U}^\bullet\mathrm{M}$ be any locally defined Fuchsian filtration. By Theorem 2.2, $\mathrm{Gr}_\mathrm{U}\mathrm{M}$ is a holonomic $\mathcal{D}_{[\mathrm{T_Z X}]}$-module.

Note that the endomorphism

$$\mathrm{E}\colon \mathrm{Gr}_\mathrm{U}^\bullet\mathrm{M} \to \mathrm{Gr}_\mathrm{U}^\bullet\mathrm{M}, \quad \mathrm{Gr}_\mathrm{U}^k\mathrm{M} \ni v \mapsto \theta(v) - kv$$

of $\mathrm{Gr}_\mathrm{U}^\bullet\mathrm{M}$ is $\mathcal{D}_{[\mathrm{T_Z X}]}$-linear. Since $\mathrm{Gr}_\mathrm{U}^\bullet\mathrm{M}$ is holonomic, the $\mathcal{D}$-linear endomorphism space is finite dimensional (locally on Z). Therefore there exists $b \in \mathbb{C}[s]$, locally on Z, such that $b(\mathrm{E}) = 0$ identically. This translates back to $b(\theta - k)\mathrm{U}^k\mathrm{M} \subset \mathrm{U}^{k+1}\mathrm{M}$, i.e., M is specializable. Since the canonical filtration is Fuchsian, we conclude that $\nu_\mathrm{Z}\mathrm{M}$ is a holonomic $\mathcal{D}_{[\mathrm{T_Z X}]}$-module.