1. Let $\mathbb{N}$ be the category whose objects are non-negative integers $0, 1, \ldots$, with

$$\mathrm{Hom}(i,j) = \begin{cases} \{*\} & i \le j \\ \emptyset & i > j \end{cases}.$$

We equip this category with the coarsest topology, making it into a site. Then a sheaf on $\mathbb{N}$ is just an inverse system $S_\bullet$ of sets: $S_0 \leftarrow S_1 \leftarrow \cdots$. The global section functor $\Gamma(\mathbb{N}, S_\bullet)$ is nothing but the (inverse) limit functor lim. Let $F_\bullet$ be a complex of sheaves of abelian groups (i.e., an inverse system of abelian groups) on $\mathbb{N}$, define

$$\mathbb{R}\lim F_\bullet = \mathbb{R}\Gamma(\mathbb{N}, F_\bullet).$$

This is just the derived inverse limit functor. It is well-known that the cohomology groups $\mathbb{R}^j \lim$ are always zero so long as $j > 1$, and one has an explicit formula for $\mathbb{R}^1 \lim$:

$$\mathbb{R}^1 \lim F_\bullet = \mathrm{Coker}(\Delta)$$

where $\Delta : \prod_{i=0}^\infty F_i \to \prod_{i=0}^\infty F_i$ is given by the following set-theoretic formula

$$\Delta(a_0, a_1, a_2, \cdots, a_i, a_{i+1}, \cdots) = (a_0 - \bar{a}_1, a_1 - \bar{a}_2, \cdots, a_i - \bar{a}_{i+1}, \cdots).$$

Here, $\bar{a}_j$ is the image of $a_j \in F_j$ in F_{j-1} under the transition map $F_j \to F_{j-1}$. See [Wei94, §3.5] for more about the derived inverse limit functors of abelian groups.

2. Let X be a topological space. Let $U_0 \subset U_1 \subset \cdots$ be open subsets of X. Assume that $X = \bigcup_i U_i$. Let $\mathcal{F}$ be a sheaf of abelian groups. Then we can define a left exact functor $\mathrm{Shv}(X) \to \mathrm{Shv}(\mathbb{N})$

$$\mathcal{F} \mapsto (F_i)_{i=0}^\infty, \quad F_i = \Gamma(U_i, \mathcal{F}).$$

Also, since $\mathcal{F}$ is a sheaf, by gluing we have $\Gamma(X, \mathcal{F}) = \lim F_i$. Thereby the Grothendieck spectral sequence for composition of functors [SP∞, Tag 015N] gives us a spectral sequence:

$$E_2^{i,j} = \mathbb{R}^i \mathrm{lim}_m \mathrm{H}^j(U_m, \mathcal{F}) \Rightarrow \mathrm{H}^{i+j}(X, \mathcal{F}).$$

The spectral sequence has at most two nonzero columns for $i = 0$ or 1, thanks to the vanishing of higher derived inverse limit functors, as we have commented in §1 above. The spectral sequence thus degenerates to a collection of exact sequences

$$0 \to \mathbb{R}^1 \mathrm{lim}_m \mathrm{H}^{i-1}(U_m, \mathcal{F}) \to \mathrm{H}^i(X, \mathcal{F}) \to \mathrm{lim}_m \mathrm{H}^i(U_m, \mathcal{F}) \to 0.$$

3. Assume now $\mathcal{F}$ are all acyclic on U_m for all m, then we get from the exact sequence in (2) that whenever $i > 1$, $\mathrm{H}^i(X, \mathcal{F})$ is identified with the limit of $\mathrm{H}^i(U_m, \mathcal{F})$, which are all zero. Thanks to (2) the natural map

$$\mathrm{H}^1(X, \mathcal{F}) \to \mathrm{lim}_m \mathrm{H}^1(U_m, \mathcal{F})$$

is an isomorphism if and only if $\mathbb{R}^1 \lim \mathcal{F}(U_\bullet)$ is zero.

4. Let us specialize to a situation where the above general nonsense can be applied. Let $\mathfrak{o}$ be a complete discrete valuation ring with a fixed uniformizer ϖ and fraction field K. Let $X = \bigcup U_i$ be a rigid analytic space over K that is an admissible nested union of Weierstraß domains. We write $U_i = \mathrm{Sp}(A_i)$. Let $\mathcal{F}$ be a coherent sheaf on X, which is defined by a sequence of finitely generated A_i-module M_i in a way that there are isomorphisms.

$$M_i \otimes_{A_i} A_{i-1} \to M_{i-1}$$

Since $A_e \to A_{e-1}$ has dense image with respect to the ϖ-adic topology (an easy fact of Weierstraß immersions, cf. a lemma of this post), the map $\psi_e : M_e \to M_{e-1}$ also has dense image with respect to some suitable quotient topology. Since each U_i is affinoid, $\mathrm{H}^j(U_i, \mathcal{F}) = 0$ for all $j > 0$, and this implies $\mathrm{H}^j(X, \mathcal{F}) = 0$ for all $j > 1$. But more is true.

Claim. *We have* $\mathrm{H}^1(X, \mathcal{F}) = 0$ *for all* $j > 0$.

Proof. We need to prove $\mathbb{R}^1 \lim_e M_e = 0$. One useful criterion for the vanishing is the surjectivity of the transition maps. In the present situation the surjectivity does not hold, but with the help of the topology, the density of the images of the transition maps can help us to mimic the "finite proof".

First, we choose for each M_n an A_n°-lattice L_n inside M_n (A_n° being the ring of power bounded elements in A_n). Replacing L_n by $L_n \cap \psi^{-1} L_{n-1}$ we can assume that $\psi(L_{n+1}) \subset L_n$. Fix $(f_e) \in \prod M_e$. Define $g_0 = 0$. By density, there exists $g_1 \in M_1$ such that $\epsilon_0 = -\psi(g_1) + f_0 \in \varpi L_0$. Inductively, assuming $g_{i+1} \in M_i, \epsilon_i \in \varpi^{i+1} L_i$ have been defined. Then by density, we can find $g_{i+2} \in M_{i+2}$ such that $\epsilon_{i+1} = f_{i+1} + g_{i+1} - \psi(g_{i+2}) \in \varpi^{i+2} L_{i+1}$. Define

$$h_e = g_e + \sum_{i \geq e} \overline{\epsilon_i} \in M_e,$$

where $\overline{\epsilon_i}$ is the image of ϵ_i in M_e. Then

$$h_e - \psi(h_{e+1}) = g_e - \psi(g_{e+1}) + \epsilon_e = f_e,$$

thus (f_e) is cohomologous to zero. We win. □

Corollary 5. *If a rigid analytic space X is a nested union of affinoid Weierstraß subdomains, then all coherent $\mathcal{O}_X$-modules are acyclic.*

This result is due to Kihel [Kie67, Satz 2.4.2] (known as "Theorem B"). In [Kie67], a *quasi-Stein* space is defined to be a rigid analytic space that can be written as a nested admissible union of affinoid subdomains U_i such that

$$\mathcal{O}_X(U_i) \to \mathcal{O}_X(U_{i-1})$$

has dense image with respect to the nonarchimedean topology (i.e., *Runge* open immersions). It turns out that such open immersions are necessarily open immersions of Weierstraß domains, see [Bos14, 4.2/5, 4.2/7]). The rigid analytic affine space and open polydisks in the affine space are examples of quasi-Stein spaces.

6. *Unimportant comments.* Retain the notations of §4. Recall the notion of convergent topos. Let (P, Z, z) be a widening for a variety X over $\mathfrak{o}/\varpi\mathfrak{o}$. Then one constructs a colimit topos $\vec{P}^\sim$ by means of the universal enlargements. There is a natural morphism of topoi

$$\gamma : \vec{P}^\sim \to Z_{\mathrm{zar}}^\sim.$$

Let $M_\bullet$ be an *admissible* coherent sheaf on $\vec{P}$, i.e., the transition maps

$$M_n \otimes_{\mathcal{O}_{P_n}} \mathcal{O}_{P_{n-1}} \to M_{n-1}$$

are onto, then by the Corollary we conclude that

$$R\gamma_* M = \gamma_* M.$$

This incarnation of Kiehl's theorem B in convergent cohomology is important in the comparison between rigid and convergent cohomology.

References

[Bos14] — Bosch, S. (2014). *Lectures on formal and rigid geometry*. Springer, Cham.

[Kie67] — Kiehl, R. (1967). *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*. Invent. Math., 2, 256–273.

[Wei94] — Weibel, C. A. (1994). *An introduction to homological algebra*. Cambridge University Press, Cambridge.

[SP∞] — The Stacks Project Authors (est. 2009). Stacks Project.