# THOMASON-TROBAUGH LOCALIZATION VIA BOUSFIELD TECHNIQUE

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This note is devoted to describe a localization theorem of Thomason and Trobaugh. The original from is in Thomason and Trobaugh 1990. The presentation is due to Neeman 1992.

## 1. Brown representation

In this section we review Brown representation theorem, formulated in terms of triangulated categories. We do not attempt to give the most general formulation of this theorem, and are content to ourselves to the compactly generated case, referring the interested reader to Neeman 2001. Although the theory now has been classical, we include a brief outline of the proof for the sake of completeness (although we are unable to put any hint about homotopy limits and colimits, neither the definitions nor their properties, in our outline of proof).

Recall that an object $c$ in a triangulated category is called *compact* if $\mathrm{Hom}(c, \bullet)$ commutes with arbitrary small colimits existed in the category. Recall also a triangulated category $\mathscr{D}$ is called *compactly generated* provided it has a small set of compact (weak) generators, that is there are objects $k_i \in \mathscr{D}$ which are compact, such that $\mathrm{Hom}_{\mathscr{D}}(k_i, T) = 0$ for all $i$ implies $T = 0$.

Also, suppose $\mathscr{D}$ is a triangulated category with small coproducts. Recall that a triangulated subcategory $\mathscr{S}$ of $\mathscr{D}$ is called *localizing*, if the inclusion functor preserves arbitrary small coproducts.

For a triangulated category with small coproducts, any reasonable cohomological functor is representable. This is known as Brown's representability theorem. If we take the triangulated category to be the stable homotopy category of spectra, then it is visible that the sphere spectrum $\mathbb{S}$ is a compact generator. The usual topological Brown theorem is a specialization of the general theorem concerning triangulated category. The proof of the generalized Brown theorem is almost identical to the classical topological one, and we briefly outline it.

***Theorem* 1.1** (Neeman 1996, Theorem 3.1)**.** *Let $\mathscr{D}$ be a compactly generated triangulated category. Suppose $\mathscr{D}$ has arbitrary small coproducts. Let $H$ be a cohomological functor on $\mathscr{D}$ with values in the category of abelian groups. Suppose $H$ takes coproducts into products. Then $H$ is representable.*

*Sketch of proof.* The proof follows the following outline. First, we "approximate" the functor $H$ by a sequence of representable functors. To be precise, we build a sequence of

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objects $X_i$, such that we have the following diagram of functors:

$$
\begin{array}{ccccccc}
K_1 & \longrightarrow & K_2 & \longrightarrow & \dots & \longrightarrow \\
\downarrow & & \downarrow & & & \\
X_1 & \longrightarrow & X_2 & \longrightarrow & \dots & \longrightarrow \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & \\
H & \longrightarrow & H & \longrightarrow & \dots & \longrightarrow
\end{array}
\tag{1.2}
$$

with the following properties:

(1) if we evaluate the functors at the set of compact generators, each map $K_i \to K_{i+1}$ is the zero map;
(2) if we evaluate the functors at the set of compact generators, each map $f_i$ is surjective.

Next, we take the homotopy colimit of the sequence $X_1 \to X_2 \to \cdots$, and denote the limit by $X$. We also denote the set of compact generators by $\{S_i : i \in I\}$. Since $S_i$ are all compact, we see that for each $i$,

$$\mathrm{Hom}(S_i, X) = \lim \mathrm{Hom}(S_i, X_j) = H(S_i).$$

Hence, $\mathrm{Hom}(\bullet, X)$ coincide with $H$ on the set of compact generators.

Now we form the subcateogry $\mathscr{A}$ consists all objects of $\mathscr{D}$ on which the natural transformation $h_X \Rightarrow H$ is an isomorphism for all cohomological functors commuting with coproducts. Then it's easily shown that $\mathscr{A}$ contains the set of compact generators, and is localizing. Moreover, the Brown representation theorem holds for $\mathscr{A}$. Using standard technique (see Lemma 2.2 below), we see the inclusion functor $\mathscr{A} \to \mathscr{D}$ has a right adjoint, so each object in $X$ fits a distinguished triangle

$$rX \to X \to \ell X \to rX[1],$$

where $\ell X$ is in $\mathscr{A}$ and $rX$ is perpendicular to $\mathscr{A}$. However since $\mathscr{A}$ contains the compact generators, it follows that $rX$ is trivial, implying $X \cong \ell X$, hence $X \in \mathscr{A}$ and $\mathscr{A}$ is equivalent to $\mathscr{D}$. Thus Brown representation also holds for $\mathscr{D}$.

It remains to build the approximation (1.2). This technique should be familiar for topologists who is acquitant with the classical Brown representation found in textbooks. We build the gadgets $X_i$ step by step. To start with, let

$$X_1 = \bigoplus_{i \in I} \bigoplus_{g \in H(S_i)} S_i,$$

then we see

$$H(X_1) = \prod_{i \in I, g \in H(S_i)} H(S_i).$$

There is a canonical element in $H(X_1)$, namely $(g : i \in I, g \in H(S_i))$. This gives rise a natural transformation $f_1 : X_1 \Rightarrow H$, where by abuse of notation we also use $X_1$ to denote the functor $\mathrm{Hom}(-, X_1)$. Now the mapping $f_1$, evaluated at $S_i$ simply gives a projection to $H(S_i)$, and whence surjective. Let $K_1$ be the kernel functor. Thus we have constructed the first vertical line in (1.2).

The preceding construction is similar to the topological construction of attaching cells, keeping in mind of the analogy between compact generator and sphere spectrum. Now the next step is to attach more cells to kill the term $K_1$.

Replacing $H$ be $K_1$, we may build a space $Y_1$, and a map $Y_1 \to X_1$. Let $X_2$ be the homotopy cokernel of this mapping. Using the fact that $K_1 \to X_1 \to H$ is zero, we see the mapping $X_1 \to H$ factors through $X_2$. We could also do the same thing for $X_2 \to H$ and build all the forthcoming $X_n$. This finishes the proof of the theorem. □

Brown's theorem has lots of interesting consequences. For instance, the duality theorem of Grothendieck is almost a trivial consequence of it (knowing that a reasonably nice scheme has compactly generated derived category of quasi-coherent sheaves). However, we will only use this theorem in some simple situations.

***Remark* 1.3** (Failure of Brown representation in general)**.** The Brown representation theorem actually holds true for the derived categories of Grothendieck categories (See, e.g., Alonso Tarrıo, Jeremıas López, and Souto Salorio 2000). However it may be false if the axiom asserting the existence of generators is dropped. Let's consider the following example due to Dan Freyd. Let $\mathscr{A}$ be the abelian category whose objects are abelian groups, together with a collection $\varphi_i$ of endomorphisms, one for each small cardinal $i$, and whose arrows are maps of abelian groups commuting with these $\varphi_i$. This is an abelian category with arbitrary small limits and colimits. However its derived category is by no means a category (rather, it's merely a portly category). In fact, if $\mathbb{Z}$ is the object whose ambient group is $\mathbb{Z}$ and all endomorphisms are zero, then the extension class

$$\mathrm{Ext}_{\mathscr{A}}(\mathbb{Z}, \mathbb{Z})$$

is a proper class: since for each small cardinal $i$ one can construct an extension class $M_i$ and different $M_i$ are pairwise nonisomorphic. In fact, one defines $M_i$ to be the group $\mathbb{Z} \oplus \mathbb{Z}$ and $\varphi_i$ is defined by the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and all $\varphi_j$ are zero if $j \neq i$. In this case, the inclusion functor from the triangulated category of acyclic complexes into that all complexes has neither left nor right adjoint, although it preserves arbitrary small limits and colimits, the reason is simple: if either adjoint exists, the derived category would isomorphic to a full triangulated subcategory of $K(\mathscr{A})$, which is small. This contradicts to what we have seen above: the derived category of $\mathscr{A}$ is merely a portly category. See Casacuberta and Neeman 2009 for a more detailed discussion.

## 2. Bousfield localization

We take some pages to review the Bousfield localization theorem, formulated in terms of triangulated categories.

***Definition* 2.1.** Let $\mathscr{D}$ be a triangulated category with small coproducts. A subcategory $\mathscr{S}$ of $\mathscr{D}$ is called *localizing* if for any index set $I$, any collection of objects $\{S_i : i \in I\}$ of $\mathscr{S}$, the coproduct $\bigoplus_{i \in I} S_i$ is inside $\mathscr{S}$.

Localizing subcategories are usually defined to be those whose inclusion functor has a right adjoint. It turns out when $\mathscr{S}$ is compactly generated this is equivalent to the inclusion functor preserves coproducts. See Lemma 2.2 below.

***Lemma* 2.2.** *Let $\mathscr{D}$ be a triangulated category with small coproducts. Let $\mathscr{S}$ be a localizing subcategory of $\mathscr{D}$ weakly generated by a small set of compact objects. Then the inclusion functor $i : \mathscr{S} \to \mathscr{D}$ has a right adjoint.*

*Proof.* This is a conclusion of the general Brown representation theorem of compactly generated triangulated categories. Consider for each $X \in \mathscr{D}$ the functor $\mathrm{Hom}_{\mathscr{D}}(i(\bullet), X)$. Then this is a cohomological functor taking coproducts into products (thanks to the fact that $\mathscr{S}$ has small coproduct and $i$ preserves coproducts). By Brown representation theorem, Theorem 1.1, this functor is represented by an object $pX$ in $\mathscr{S}$. It follows that the assignment $X \mapsto pX$ is functorial, and that $p$ is a right adjoint to $i$. □

***Definition* 2.3.** Let $\mathscr{D}$ be a triangulated category. Let $\mathscr{S}$ be a triangulated subcategory. An object $T$ of $\mathscr{D}$ is called *localizing* with respect to $\mathscr{S}$ if for each $S \in \mathscr{S}$, we have

$$\mathrm{Hom}_{\mathscr{D}}(S, T) = 0.$$

Let $\mathscr{S}^{\perp}$ denote the category of all $\mathscr{S}$-local objects in $\mathscr{D}$.

***Lemma* 2.4.** *Let notations and assumptions be as in Lemma 2.2. Let $q : \mathscr{D} \to \mathscr{D}/\mathscr{S}$ be the Verdier quotient functor. Then the composition $\mathscr{S}^{\perp} \to \mathscr{D} \to \mathscr{D}/\mathscr{S}$ is an equivalence of triangulated categories. Moreover, if $j$ is a quasi-inverse to the equivalence, then $j$ is right adjoint to $q$.*

*Proof.* This follows from the assertion that for each $T \in \mathscr{S}^{\perp}$, and each $X \in \mathscr{D}$, we have

$$\mathrm{Hom}_{\mathscr{D}}(X, T) = \mathrm{Hom}_{\mathscr{D}/\mathscr{S}}(qX, qT).$$

The proof of the assertion is just a play of calculus of fractions in triangulated categories and we omit it. □

We may draw some conclusions from the principles above.

***Lemma* 2.5.** *Let $\mathscr{D}$ be triangulated category weakly generated by a small set $R$ of compact objects. Suppose that $\mathscr{D}$ has small coproducts. Then the smallest localizing triangulated subcategory of $\mathscr{D}$ containing $R$ is $\mathscr{D}$ itself.*

*Proof.* Let $\mathscr{A}$ be the smallest localizing subcategory of $\mathscr{D}$ containing $R$. Now the Bousfield theory can be applied to the inclusion $\mathscr{A} \subset \mathscr{D}$, and Lemma 2.2 produces a right adjoint functor $\ell : \mathscr{D} \to \mathscr{A}$ to the inclusion functor. It follows that any object $X \in \mathscr{D}$ fits into a distinguished triangle

$$rX \to X \to \ell X \to rX[1]$$

where $rX$ is in $\mathscr{A}^{\perp}$. But since $R$ is a set of generators, we must have $rX = 0$. Hence $X \cong \ell X$. This shows $\mathscr{A} = \mathscr{D}$. □

***Lemma* 2.6** (Ravenel)**.** *Let notations be as in Lemma 2.4. The category $\mathscr{S}^{\perp}$ is localizing. Hence the functor $j$ preserves coproducts.*

*Proof.* Let $X_i$ be a collection of objects in $\mathscr{S}^{\perp}$. We shall prove $X = \bigoplus_i X_i$ is also an $\mathscr{S}$-local object. Let $\mathscr{A}$ be the smallest triangulated subcategory of $\mathscr{S}$ containing objects perpendicular to $X$, i.e. those $S$ such that $\mathrm{Hom}(S, X) = 0$. To begin with, we note the set $R$

of compact generators of $\mathscr{S}$ is contained in the set $\mathrm{Ob}(\mathscr{A})$. Indeed, let $A \in R$ be compact, then we have

$$\mathrm{Hom}(A, X) = \bigoplus_{i \in I} \mathrm{Hom}(A, X_i) = 0,$$

as each $X_i$ is localizing. Moreover, it is simple to check that $\mathscr{A}$ is localizing. Thus $\mathscr{A}$ is a localizing triangulated subcategory of $\mathscr{S}$ containing $R$. Since $R$ is already a set of compact generator of $\mathscr{S}$, it generates $\mathscr{A}$ weakly too. Now we may apply Lemma 2.5 to conclude the proof of Ravenel's lemma. □

## 3. Thomason-Trobaugh localization

This section is devoted to proof a theorem of Thomason and Trobaugh, Theorem 3.2, following the method of Neeman.

We assume the following situation.

**3.1.** Let $\mathscr{D}$ be a triangulated category with arbitrary coproducts. Suppose $\mathscr{D}$ is the smallest localizang subcategory containing $\mathscr{D}^c$. Let $R$ be a small set of compact objects of $\mathscr{D}$. Let $\mathscr{S}$ be the smallest localizing subcategory containing $R$. Form the Verdier quotient $q : \mathscr{D} \to \mathscr{D}/\mathscr{S}$. Denote the quotient category by $\mathscr{T}$.

Recall that a triangulated subcategory $\mathscr{S} \subset \mathscr{D}$ of a triangulated subcategory is *saturated* or *epaisse*, if every direct summand of an object $S \in \mathscr{S}$ is also in $\mathscr{S}$. We shall always assume a saturated subcategory to be strictly full. For any subset $R$ of $\mathscr{D}$, the *saturation*, or *epaisee closure*, of $R$ is the smallest strictly full, saturated, triangulated subcategory containing $R$.

The main theorem is the following.

***Theorem* 3.2.** *In the situation above, we have the following.*

(1) *The inclusion functor $\mathscr{S} \to \mathscr{C}$ takes $\mathscr{S}^c$ to $\mathscr{D}^c$;*
(2) *$\mathscr{S}^c$ is the saturation of of $R$.*
(3) *the quotient functor $q$ takes $\mathscr{D}^c$ to $\mathscr{T}^c$,*
(4) *the natural functor $f : \mathscr{D}^c/\mathscr{S}^c \to \mathscr{T}^c$ is fully faithful, and*
(5) *$\mathscr{T}^c$ is the saturation of the image of $f$.*

The proof of Theorem 3.2 will occupy the rest of this section.

*Proof of Theorem 3.2, Item (1).* Let $X$ be a compact object in $\mathscr{S}$. We shall prove it is also comapct in $\mathscr{D}$. To this end, let $X_i$ be a collection of objects in $\mathscr{D}$. We would like to prove that

$$\mathrm{Hom}_{\mathscr{D}}(X, \bigoplus X_i) = \bigoplus \mathrm{Hom}(X, X_i).$$

Now each $X_i$ can be put into a distinguished triangle

$$rX_i \to X_i \to \ell X_i \to rX_i[1],$$

where $\ell$ is the colocalization functor to $\mathscr{S}$ (right adjoint to the inclusion), and $r$ the Bousfield localization to the complement $\mathscr{S}^{\perp}$. Applying $\mathrm{Hom}_{\mathscr{D}}(X, -)$, we get an isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(X, X_i) = \mathrm{Hom}_{\mathscr{S}}(X, \ell X_i). \tag{3.3}$$

Using Lemma 2.6, we see the direct sum $\oplus r X_i$ is also in $\mathscr{S}^{\perp}$. Now we take the direct sum of all these distinguished triangles, yielding a distinguished triangle

$$\bigoplus rX_i \to \bigoplus X_i \to \bigoplus \ell X_i \to \bigoplus rX_i[1]$$

(it's an exercise to prove the coproduct of distinguished triangles is also distinguished). Applying the functor $\mathrm{Hom}(X, -)$ to the distinguished triangle, and using that $\bigoplus rX_i$ is perpendicular to $X$, we have

$$\mathrm{Hom}_{\mathscr{D}}(X, \bigoplus X_i) \cong \mathrm{Hom}_{\mathscr{D}}(X, \bigoplus \ell X_i).$$

Since each $\ell X_i$ is in $\mathscr{S}$, and since $\mathscr{S}$ is localizing, we see the right hand side of the equation equals $\bigoplus \mathrm{Hom}_{\mathscr{S}}(X, \ell X_i)$. Now we can use the equation (3.3) to conclude that $X$ is compact. □

The proof of Item (1) can be generalized a little bit.

***Proposition* 3.4.** *Let $F : \mathscr{D} \to \mathscr{T}$ be an exact functor between triangulated categories. Suppose that*

(1) *$F$ has a right adjoint $G$, and*
(2) *$\mathscr{D}$ is compactly generated.*

*Then $F$ takes compact objects to compact objects if and only if $G$ preserves small coproducts.*

*Proof.* The proof of Item (1) is also adapted to the "if" direction. Note that Ravenel's Lemma 2.6 is where we get that $G$ preserves compact objects.

To show the "only if" direction, suppose $F$ takes compact objects to compact objects. Let $\{Y_i : i \in I\}$ be a small set of objects in $\mathscr{T}$. We need to prove the natural arrow $\varphi : G \bigoplus Y_i \to \bigoplus GY_i$ is an isomorphism. Let $\mathscr{D}'$ be the smallest full triangulated subcategory of $\mathscr{D}$ consisting objects $X$ such that the mapping $\varphi_X : \mathrm{Hom}(X, G \bigoplus Y_i) \to \bigoplus \mathrm{Hom}(X, GY_i)$ is an isomorphism. Then, by manipulating with adjunctions, we see all compact objects are contained in $\mathscr{D}'$. Also, $\mathscr{D}'$ is visibly localizing. Using Lemma 2.5, we see $\mathscr{D}' = \mathscr{D}$. □

The proof of Item (2) of Theorem 3.2 is reduced to the following lemma.

***Lemma* 3.5.** *Let $\mathscr{D}$ be a compactly generated triangulated category with small coproducts. Suppose $R$ is a small set of compact generators. Then $\mathscr{D}^c$ is the smallest strictly full, saturated, triangulated subcategory containing $R$.*

*Proof.* We introduce some notations. Let $\langle R\rangle_n$ be the strictly full subcategory of $\mathscr{D}$ obtained by taking finite direct sums and summands of cones of arrows with domain of codomain lying in $\langle R\rangle_1$ and $\langle R\rangle_{n-1}$, with $\langle R\rangle_1$ equal to the full subcategory generated by finite sums of objects in the set $R$. We use $[R]_n$ to denote the similar subcategories, with finite sums replaced by infinite sums. Let $X$ be any compact object in $\mathscr{D}$, it suffices to prove $X$ lies in $\langle R\rangle_n$ for some $n \in \mathbb{Z}_{\geq 0}$.

By Lemma 2.5, we have $\cup[R]_n = \mathscr{D}$. Thus $X$ lies in $[R]_n$ for some $n$. Therefore there exists $X'$ in $\mathscr{D}$, such that $X \oplus X'$ fits a distinguished triangle:

$$Z_{n-1} \to X \oplus X' \to M_{n-1}.$$

where $M_{n-1} \in [R]_{n-1}$ and $M_{n-1} \in [R]_1$. Consider the composition $X \to X \oplus X' \to M_{n-1}$. Since $X$ is compact, we see it factors through some compact object $Y_1$ by the virtue of that $M_{n-1} \in [R]_0$. Let $X_{n-1}$ be the homotopy kernel of $X \to Y_1$. Then $X_{n-1}$ is also compact.

Since $Z_{n-1}$ is in $[R]_{n-1}$, there exists a distinguished triangle

$$Z_{n-2} \to Z_{n-1} \oplus Z'_{n-1} \to M_{n-2} \to$$

and we have a mapping $X_{n-1} \to Z_{n-1} \oplus Z'_{n-1}$. We could play the same game as we did for $X$. This will yield a distinguished triangle of compact objects

$$X_{n-2} \to X_{n-1} \to Y_2 \to$$

with $Y_2 \in \langle R \rangle_1$. Continuing this process, we get a sequence

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & X_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_{n-1} & \longrightarrow & Z_n \end{array}$$

such that

(1) the cone of $X_i \to X_{i-1}$ is in $\langle R \rangle_1$, and
(2) $Z_i$ lies in $[R]_i$.

It follows from the octahedral axiom that the cone $M$ of $X_0 \to X_n = X$ lies in $\langle R \rangle_n$. Consider the diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\alpha} & X & \longrightarrow & M & \longrightarrow & \\ \downarrow & & \downarrow & & & & . \\ 0 & \longrightarrow & X \oplus X' & & & & \end{array}$$

Since the middle vertical arrow splits, it follows that $\alpha = 0$. Therefore $X$ is a summand of $M$, which is in $\langle R \rangle_n$, as desired. □

*Proof of Theorem 3.2, Item (3).* By Bousfield localization, we could identify $\mathcal{T}$ with the category $\mathcal{S}^\perp$. We see the Verdier quotient functor $q$ is identified with the right projection $r$. By adjunction, we have for each $X$ and each collection $Y_i$ in $\mathcal{S}^\perp$,

$$\mathrm{Hom}_{\mathcal{D}}(X, j(\oplus Y_i)) = \mathrm{Hom}_{\mathcal{S}^\perp}(rX, \oplus Y_i).$$

Since $\mathcal{S}^\perp$ is localizing (Lemma 2.6), the inclusion functor $j$ preserves coproducts. Hence if $X$ is compact, we also have $qX$ compact. □

*Proof of Theorem 3.2, Item (4).* We shall prove the functor $\mathcal{D}^c/\mathcal{S}^c \to \mathcal{T}^c$ is fully faithful. This follows from a similar argument as in Lemma 3.5. We leave it to the reader. □

*Proof of Theorem 3.2, Item (5).* Recall that we have assumed that $\mathcal{D}^c$ is essentially small. Let $S$ be a collection of objects in $\mathcal{D}^c$ containing all isomorphism classes of objects. Then $S$ form a set of compact generators of $\mathcal{D}$. By Item (3), the image $qS$ in $\mathcal{T}$ are compact in $\mathcal{T}$. Moreover, $qS$ is a set of compact generators of $\mathcal{T}$. Suppose $T \in \mathcal{T}$ is perpendicular to all $qK$, $K \in S$, then we have

$$\mathrm{Hom}_{\mathcal{D}}(K, jS) = \mathrm{Hom}_{\mathcal{T}}(qK, S) = 0$$

for all $K \in S$. It follows that $jS$, whence $S$, equals zero. Using Lemma 3.5, we conclude that $\mathscr{T}^c$ is the saturation of the image of $\mathscr{D}^c$ in $\mathscr{T}$. □

Now we go to the geometric context to see some interesting application of Theorem 3.2.

Let $X$ be a quasi-compact, separated scheme. Let $\mathrm{D}(X)$ be the derived category of quasi-coherent sheaves on $X$. Let $U \subset X$ be an open subscheme. Let $Z$ be the complement of $U$. One basic question of K-theory is to relate the K-theories of $X$, $U$ and $Z$. Let $\mathscr{S}$ be the strictly full, saturated, triangulated subcategory of $\mathrm{D}(X)$ consisting complexes whose cohomology sheaves are supported in the scheme $Z$.

***Lemma* 3.6.** *Let notations be as above. There is a natural equivalence*

$$\mathrm{D}(X)/\mathscr{A} \to \mathrm{D}(U)$$

*of triangulated categories.*

*Proof.* Let $E$ be a complex on $X$ whose cohomology is supported in $Z$. Then the restriction of $E$ to $U$ has no cohomology, by the virtue of flat base change property. Hence any object in $\mathscr{A}$ is mapped to zero. This gives a natural triangulated functor $\Phi : \mathrm{D}(X)/\mathscr{A} \to \mathrm{D}(U)$. This functor is essentially surjective, by the virtue that quasi-coherent sheaves on $U$ can always be extended to whole $X$, see Stacks Project, Lemma 01PE (1) (noting that we have assumed that $U$ is quasi-compact).

Let's show this functor is fully faithful. Let $E$ and $F$ be two complexes of quasi-coherent sheaves on $X$. Suppose we have a roof $(G, s, f) : E|_U \to F|_U$ in the derived category $\mathrm{D}(U)$. Using Stacks Project, Lemma 01PE, (3), we could get a roof $(H, u, g)$, where $H$ is a complex of quasi-coherent sheaves restricts to $G$, $u : H \to E$ restricts to $s$, and $g : H \to F$ restricts $f$. Since $s$ is an quasi-isomorphism, as complexes on $U$, the cohomology of the mapping cone of $u$ has cohomology lying in $Z$. Therefore $\Phi$ is a full functor.

Now suppose $(H, u, g)$ is a roof from $E$ to $F$, such that the cone of $u$ has cohomology supported on $Z$. Then we show this roof is equivalent to the trivial roof with respect to $\mathscr{A}$. Indeed, restricting to $U$ we see $u$ induces a quasi-isomorphism, hence the map $g|_U$ is zero in $\mathrm{D}(U)$. By standard technique of triangulated categories, we could replace $G|_U$ by some complex and $G$ by the extension of that complex, such that $g|_U$ is honestly zero. Thus the cone of $g$ is isomorphic to $F$ upon localizing on $U$. Thus $G$ has cohomology sheaves supported on $Z$, proving that $\Phi$ is faithful. □

Now, the categories $\mathrm{D}(X)$ and $\mathrm{D}(U)$ are compactly generated triangulated categories. This is proved by Thomason-Trobaugh when $X$ carries an amply family of line bundles, and is proved by A. Neeman, in the lack of line bundles, in the beautiful paper Neeman 1996 about Grothendieck duality, assuming $X$ is quasi-compact and separated. The most general known form is probably the one in Bondal and Van den Bergh 2003, in which the authors showed that $\mathrm{D}_{\mathrm{QCoh}}(X)$ is compactly generated when $X$ is quasi-compact and quasi-separated. In fact, compact objects in these categories are precisely perfect complexes on $X$ and $U$, and those supported in $Z$. Applying Theorem 3.2, we get a fully faithful functor $\mathrm{Perf}(X)/\mathscr{A}^c \to \mathrm{Perf}(U)$. Upon taking saturation, these two triangulated categories are just the same.

The highlight of the theory of Thomason-Trobaugh is that the K-theory of $X$ and $U$ may be computed purely in terms of perfect complexes, i.e., in terms of the K-theory of the triangulated categories Perf($X$) and Perf($U$). Thus, if the K-theory can be suitably defined for triangulated categories, and is functorial with respect to Verdier quotient, then the above result will imply a strong result about K-theory spectra. However, the definitions of K-theories rely on various choices of "models".

***Theorem* 3.7** (Thomason-Trobaugh)**.** *Let $X$ be a scheme that is quasi-compact and separated. Let $U$ be a quasi-compact, open subscheme of $X$. Let $\mathcal{C}$ be the Waldhausen category whose objects are perfect complexes on $X$ with cohomology sheaves lying in $X \setminus U$. Then there is an almost fibration of K-theory spectra:*

$$K(\mathcal{C}) \to K(X) \to K(U),$$

*where $K(\mathcal{C})$ is the higher K-theory spectrum of the Waldhausen category $\mathcal{C}$.*

Here is one subtlety in the statement of the theorem: we have used the term "almost". This means that the displayed sequence is not generally a fiberation. It is only restricts to a connected component of $K(U)$. The subtlety lies in the fact that Perf$(U)$ is not quite Perf$(X)/\mathcal{A}$ — there is an issue of straightening the saturation. On the level of K-theory, this should be interpreted as that $K(\mathrm{Perf}(X)/\mathcal{A})$ is a covering space of $K(U)$.

We also remark that taking higher K-theory spectra depends on the choice of "models" of the triangulated category. In that it's not an invariant of the triangulated category itself. Neeman has shown that there exist two Waldhausen categories with equivalent homotopy categories while their K-theories are different. Therefore, the path to the conclusions expected in the above paragraph is far from trivial.

In practice, we need to lift the Verdier quotient to the model realizations, such that the triangulated functors lift to exact functors among model categories. Then we use Waldhausen's approximation and fibration theorems to deduce the desired K-theory fibration sequence.

## 4. On the epaisee closure

> . . . Tom Trobaugh, quite intelligent, singularly original, and inordinately generous, killed himself consequent to endogenous depression. Ninety-four days later, in my dream, Tom's simulacrum remarked, "*The direct limit characterization of perfect complexes shows that they extend, just as one extends a coherent sheaf*." Awaking with a start, I knew this idea had to be wrong, since some perfect complexes have a non-vanishing $K_0$ obstruction to extension. I had worked on this problem for 3 years, and saw this approach to be hopeless. But Tom's simulacrum had been so insistent, I knew he wouldn't let me sleep undisturbed until I had worked out the argument and could point to the gap. This work quickly led to the key results of this paper. . .
>
> — Robert W. Thomason

This section is to expose the proof of a technical result, see Theorem 4.1. The result appeared in the classic Thomason and Trobaugh 1990, Proposition 5.5.4; in Neeman 1992, Corollary 0.9; and in Bondal and Van den Bergh 2003, Proposition 3.3.2. Neeman's proof uses his theory of K-theory of triangulated categories. The proof presented below is the one from Bondal and Van den Bergh 2003.

***Theorem* 4.1.** *Let $\mathscr{B}$ be a triangulated category. Let $\mathscr{A}$ be a strictly full, triangulated subcategory of $\mathscr{B}$. Then an object $B$ of $\mathscr{B}$ is in $\mathscr{A}$ if and only if $[B]$ is in the image of the map $K_0(\mathscr{A}) \to K_0(\mathscr{B})$ of Grothendieck groups.*

The proof of the theorem is at the end of this section. An immediate consequence of the theorem is the following.

***Corollary* 4.2.** *Let hypotheses be as in Theorem 3.5. Then For any $B \in \mathscr{B}$, $B \oplus B[1] \in \mathscr{A}$.*

*Proof of the Corollary.* Since the class of $B \oplus B[1]$ is zero in the Grothendieck group, it lies in the image of $K(\mathscr{A})$. In the view of Theorem 4.1, we must have $B \oplus B[1] \in \mathscr{A}$. □

**4.3.** *Notations.* The rest of this section is devoted to the proof of Theorem 4.1. To this end, let's set up some notations.

- Let $F(\mathscr{A})$ be the free abelian group generated by the abelian monoid $(\mathrm{sk}(\mathscr{A}), \oplus)$, where $\mathrm{sk}(\mathscr{A})$ is the set of all isomorphism classes of $\mathscr{A}$. Similarly one defines $F(\mathscr{B})$.
- The image of $X \in \mathrm{Ob}(\mathscr{A})$ in $F(\mathscr{A})$ is denoted by $[X]$.
- Let $E(\mathscr{A})$ be the subgroup of $F(\mathscr{A})$ generated by the elements
$$[X \oplus Y] - [X] - [Y].$$
Similarly one defines $E(\mathscr{B})$.
- Let $G(\mathscr{A})$ be the quotient $F(\mathscr{A})/E(\mathscr{A})$. Similarly one defines $G(\mathscr{B})$.
- Since $X \to X \oplus Y \to Y \to$ is a distinguished triangle, there is a natural map $G(\mathscr{A}) \to K_0(\mathscr{A})$. Let $I(\mathscr{A})$ be the kernel of this map. Similarly one defines $I(\mathscr{B})$.

***Lemma* 4.4.** *Suppose $X$ and $Y$ are two objects in $\mathscr{B}$. Then $[X] = [Y]$ in $G(\mathscr{B})$ holds true if and only if there is an object $Z$ in $\mathscr{B}$ such that $X \oplus Z \cong Y \oplus Z$.*

*Proof.* The proof is easy. We omit it. □

***Lemma* 4.5.** *We have $E(\mathscr{B}) \cap F(\mathscr{A}) = E(\mathscr{A})$.*

*Proof.* Let $v \in E(\mathscr{B}) \cap F(\mathscr{A})$. Then the image of $v$ in $G(\mathscr{A})$ can be written as $[X] - [Y]$ for some $X$ and $Y$ in $\mathscr{A}$. Thus there is $w \in E(\mathscr{A})$ such that $v = [X] - [Y] + w$ in $F(\mathscr{A})$. Since $[v] = 0$ in $G(B)$, we have $[X] = [Y]$ in $G(\mathscr{B})$. It follows that there is an object $Z$ of $\mathscr{B}$ such that $X \oplus Z \cong Y \oplus Z$. Adding both sides by $Z'$, where $Z'$ is some object such that $Z \oplus Z' \in \mathscr{A}$, we may assume that $Z \in \mathscr{A}$. Therefore we have $Y \oplus Z \cong X \oplus Z$ are in $\mathscr{A}$, and it follows that

$$\begin{aligned} & v + [Y] = w + [X] \\ \Rightarrow & v + [Y \oplus Z] = w + [X \oplus Z] \\ \Rightarrow & v = w \end{aligned}$$

in $F(\mathscr{A})$. This proves the lemma. □

***Lemma* 4.6.** *In the notation above, the following hold.*

1. *The homomorphism $G(\mathscr{A}) \to G(\mathscr{B})$ is injective.*
2. *The homomorphism $I(\mathscr{A}) \to I(\mathscr{B})$ is an isomorphism.*
3. *The homomorphism $K_0(\mathscr{A}) \to K_0(\mathscr{B})$ is injective.*

*Proof.* Proof of (1). This is a formal consequence of Lemma 4.5. To see this we make a diagram chase: if some class $[v] \in G(\mathscr{A})$ is sent to zero in $G(B)$, then any preimage $v \in F(\mathscr{A})$ is sent to $E(\mathscr{B}) \subset F(\mathscr{B})$. Since $F(\mathscr{A}) \cap E(\mathscr{B}) = E(\mathscr{A})$ by Lemma 4.5, it follows that $[v] = 0$ in $G(\mathscr{A})$.

Proof of (2). Using (1), we see the map in (2) is already injective. We must show $I(\mathscr{A}) \to I(\mathscr{B})$ is surjective. By the snake lemma, this amounts to proving that

$$D(\mathscr{A})/E(\mathscr{A}) \to D(\mathscr{B})/E(\mathscr{B})$$

is surjective. In plain English, this means that we have to prove the following. For any $w \in F(\mathscr{B})$ whose $K_0$-class is zero, there exists a $w' \in F(\mathscr{B})$ that is $G$-equivalent to zero, such that $w - w'$ falls in the image of $D(\mathscr{A})$, i.e., there is an element $v \in F(\mathscr{A})$ that is $K_0$-equivalent to zero in $\mathscr{A}$, and the image of $v$ equals $w - w'$ in $F(\mathscr{B})$.

To prove this, we note that we can assume $w$ is of the form $X + Z - Y$ where $X$, $Y$ and $Z$ fit into a distinguished triangle

$$X \to Y \to Z \to X[1]$$

in $\mathscr{B}$. We want to add elements in $E(\mathscr{B})$ on the class $X + Z - Y$, so that we could transform it into $X' - Y' + Z'$ with $X'$, $Y'$ and $Z'$ in $\mathscr{A}$ and fit into a distinguished triangle. To this end, we know that there is $X_1 \in \mathscr{B}$ such that the direct sum $X \oplus X' \in \mathscr{A}$, and there is $Y_1$ such that the direct sum $Y_1 \oplus X_1 \oplus Y \in \mathscr{A}$. Now consider the trivial distinguished triangle

$$X_1 \to X_1 \oplus Y_1 \to Y_1 \to X_1[1].$$

Taking direct sum with the original triangle $X \to Y \to Z \to X[1]$, and setting $X' = X \oplus X_1$, $Y' = Y \oplus Y_1 \oplus X_1$, $Z' = Z \oplus Y_1$, we get a distinguished triangle

$$X' \to Y' \to Z' \to X'[1]$$

with $X'$ and $Y'$ are in $\mathscr{A}$. Since $\mathscr{A}$ is a triangulated subcategory of $\mathscr{B}$, $Z' \in \mathscr{A}$ too. To summarize, we have proved that

$$(X - Y + Z) + (X_1 \oplus Y_1 - X_1 - Y_1) = (X' - Y' + Z').$$

Hence the claim has been established and the (2) is proven. Finally, (3) follows (1) and (2) by the snake lemma. □

*Proof of Theorem 4.1.* Let $X$ be an object in $\mathscr{B}$ such that the K-theory class $[X]_K$ of $X$ falls in the image of $K_0(\mathscr{A})$. Then the G-class $[X]_G$ is sent to zero under the composition

$$G(\mathscr{B}) \to K_0(\mathscr{B}) \to K_0(\mathscr{B})/K_0(\mathscr{A}).$$

However, by the snake lemma, and Lemma 4.6, (1), we infer that $[X]$ lies in the image of $G(\mathscr{A}) \to G(\mathscr{B})$. It follows that we can write $[X] = [A_1] - [A_2]$ for objects $A_1$, $A_2$ of $\mathscr{A}$. By Lemma 4.4, there exists $Z \in \mathscr{B}$ such that

$$X \oplus A_2 \oplus Z \cong A_1 \oplus Z.$$

Since $\mathscr{B}$ is the epaisee closure of $\mathscr{A}$, there is $Z'$ such that $Z \oplus Z' \in \mathscr{A}$. Set $U = X \oplus A_2 \oplus Z \oplus Z'$ and $V = A_2 \oplus Z \oplus Z'$. Then $U$ and $V$ are in $\mathscr{A}$. Since there is a distinguished triangle

$$X \to U \to V \to X[1],$$

and since $\mathscr{A}$ is a triangulated subcategory, we must have $X \in \mathscr{A}$. □

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