VECTOR BUNDLES, LINE BUNDLES, PICARD GROUP

Contents

1. Vector bundles and locally free sheaves

Recall definition of a vector bundle over a scheme.

Any vector bundle $\pi\colon E \to X$ of rank r over a scheme X is determined by the following data:

- an open covering $X = \bigcup_i U_i$ such that the bundle is trivializable on U_i,
- for each pair i, j, an $r \times r$ invertible matrix g_{ij} with entries in the ring $\mathcal{O}_X(U_i \cap U_j)$,
- g_{ii} equals the identity matrix, and for each triple i, j, k, the ***cocycle condition*** $g_{jk} \cdot g_{ij} = g_{ik}$ holds (as matrices with entries in $\mathcal{O}_X(U_i \cap U_j \cap U_k)$).

These matrices g_{ij} are called ***transition functions***.

The same data determines a ***locally free*** $\mathcal{O}_X$-module $\mathcal{E}$ on X. We require $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$, and transition functions g_{ij} "glue" local pieces together.

Conversely, any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r determines a vector bundle over X, yielding transition functions g_{ij}.

From this perspective, the two notions are equivalent. In algebraic geometry, "vector bundle" and (finitely generated) "locally free sheaf" are used interchangeably.

For a vector bundle $\pi\colon E \to X$, a ***section*** is a morphism $\sigma\colon X \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_X$. For an open immersion $j\colon U \to X$, a section over U is a morphism $\sigma\colon U \to E$ such that $\pi \circ \sigma = j$. This correspondence identifies the space of sections over U with $\mathcal{E}(U)$.

To compute $\mathcal{E}(X)$ for a locally free $\mathcal{E}$ using transition functions, consider a section σ. On each trivializing open set U_i, σ induces a section σ_{U_i} of $\mathcal{E}$. The trivialization over U_i maps σ_{U_i} to an r-tuple of sections of $\mathcal{O}_X(U_i)$, i.e., a vector $\boldsymbol{v}_i$ in $\mathcal{O}_X(U_i)^{\oplus r}$. The "patching" condition is then

$$g_{ij} \cdot \boldsymbol{v}_i = \boldsymbol{v}_j$$

for all i, j.

In contrast to differential geometry, where vector bundles always have many sections, in algebraic geometry vector bundles can have many local sections but no global sections at all. See Example 3.

Working with locally free $\mathcal{O}_X$-modules has an advantage: it can be easily translated to algebra. On an affine scheme A, such a module corresponds to an A-module P. Local freeness means that there exist $f_1, \ldots, f_r \in A$, generating the unit ideal, such that $P[f_i]$ is a free $A[f_i^{-1}]$-module.

Exercise A. Let A be a ring and let P be a ***finitely generated*** A-module. Show that the following are equivalent:

- P is locally free,
- for any prime ideal $\mathfrak{p}$ of A, $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

A finitely generated locally free A-module already has a name in algebra: finitely generated projective module.

Lemma 1. *Let A be a ring. Let P be an A-module. Then the following are equivalent.*

Dec. 12, 2024.

(1) *P is a finitely generated **projective** A-module.*
(2) *P is locally free of finite rank.*

Proof. Assume (1) holds. Let us prove that $P_{\mathfrak{p}}$ is free over $A_{\mathfrak{p}}$ for any prime $\mathfrak{p}$. The module $P_{\mathfrak{p}}$ inherits finite generation from P. As P is a direct summand of a free A-module, $P_{\mathfrak{p}}$ is also a direct summand, hence projective. The quotient $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$ is thus a finite dimensional $\kappa(\mathfrak{p})$-vector space. Lifting a basis of this space to $P_{\mathfrak{p}}$ gives generators by Nakayama's lemma. Let $\alpha\colon A_{\mathfrak{p}}^{\oplus r} \to P_{\mathfrak{p}}$ be the surjection they define. The projectivity of $P_{\mathfrak{p}}$ ensures $\mathrm{Ker}(\alpha)$ is a direct summand of $A_{\mathfrak{p}}^{\oplus r}$. Since $\mathrm{Ker}(\alpha)/\mathfrak{p}\mathrm{Ker}(\alpha) = 0$, Nakayama's lemma implies $\mathrm{Ker}(\alpha) = 0$. Thus $P_{\mathfrak{p}}$ is free, proving (2).

Assume (2). Since P is locally free, it has finite presentation. For any prime $\mathfrak{p}$ and any A-module M, we have a natural isomorphism $\mathrm{Hom}_A(P, M)_{\mathfrak{p}} \simeq \mathrm{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, M_{\mathfrak{p}})$. To prove P is projective, we need to show that for any surjective map $\beta\colon M \to N$, the induced map $\beta_*\colon \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$ is surjective. The local freeness of P implies that for each prime $\mathfrak{p}$, the localized map $\beta_{\mathfrak{p}*}\colon \mathrm{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, M_{\mathfrak{p}}) \to \mathrm{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, N_{\mathfrak{p}})$ is surjective. Therefore β_* is locally surjective, hence surjective. □

Classifying finitely generated projective modules over a ring is a deep problem. A famous question posed by Serre asks whether every finitely generated projective module over $k[T_1, \dots, T_n]$ is free, where k is a field. Geometrically, this asks if all algebraic vector bundles over $\mathbf{A}_k^n$ are trivial. The affirmative answer was proved by Quillen and Suslin in 1976 independently.

A more tractable problem is to study projective modules up to ***stable equivalence***. Two projective modules P and Q over A are stably equivalent if $P \oplus A^m \simeq Q \oplus A^n$ for some m, n. The stable equivalence classes form a ring $K_0(A)$, with tensor product as multiplication and direct sum as addition. This ring $K_0(A)$ initiated the field of ***algebraic K-theory***.

Finite type locally free sheaves admit many standard operations:

- direct sums,
- tensor products,
- symmetric powers,
- exterior powers,
- dual: $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

and the behave well under (module-theoretic) inverse images.

2. Invertible sheaves and Picard group

Below, I will take a slightly different approach from standard texts like Hartshorne (1977). While Weil and Cartier divisors deserve a thorough treatment, time constraints prevent me from doing so. Yet the Picard group is too important to skip, so I offer an alternative perspective.

From now on let's focus on locally free $\mathcal{O}_X$-modules of rank one. These are known as ***invertible*** $\mathcal{O}_X$-modules, or simply ***invertible sheaves***. They correspond to vector bundles of rank one, or line bundles. Two properties are worth noting:

- the dual of an invertible sheaf is invertible,
- the tensor product of invertible sheaves is invertible.

The term "invertible" reflects a key property: for any such sheaf $\mathcal{L}$, we have $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \simeq \mathcal{O}_X$. Under the tensor product $\otimes$, isomorphism classes of invertible $\mathcal{O}_X$-modules form an abelian group with unit $\mathcal{O}_X$. This is the ***Picard group*** of X, an important algebro-geometric invariant.

Let me compute $\mathrm{Pic}(X)$ for some basic schemes. We start with:

Lemma 2. *Let A be a UFD. Then* $\mathrm{Pic}(\mathrm{Spec}(A)) = \{0\}$.

An A-module M is called invertible if its associated $\mathcal{O}_{\mathrm{Spec}(A)}$-module is an invertible sheaf. This means M is finitely generated projective of rank one. The lemma states that any invertible A-module is isomorphic to A itself.

Proof. Let I be an invertible A-module. Its dual $\mathrm{Hom}_A(I, A)$ is also invertible. Choose a nonzero ϕ in $\mathrm{Hom}_A(I, A)$. The map $\phi\colon I \to A$ must be injective, as can be verified locally.

Exercise B. Why? (This requires A to be a domain.)

So we may assume I is an ideal of A. To show $I \cong A$, we need only prove I is principal.

Pick any f in I. Since A is a UFD, we can write

$$f = u p_1^{e_1} \cdots p_r^{e_r},$$

where u is a unit and each p_i is prime.

Since I is locally free, for each i there is a principal open set $D(a_i)$ containing p_i where $I[a_i^{-1}]$ is principal. Let g_i generate $I[a_i^{-1}]$. In $A[a_i^{-1}]$ (still a UFD), write $g_i = p_i^{c_i} g_i'$ for some $c_i \geqslant 0$. As f lies in $I[a_i^{-1}]$, we have $c_i \leqslant e_i$. We claim $I = (p_1^{c_1} \cdots p_r^{c_r})$.

It suffices to show: for any a in A where $I[a^{-1}]$ is principal, $I[a^{-1}] = (p_1^{c_1} \cdots p_r^{c_r}) \cdot A[a^{-1}]$.

Exercise C. Explain why this is sufficient.

Let $I[a^{-1}] = (g)$. For primes p_j that remain non-unit in $A[a^{-1}]$, we can write

$$g = v \prod_{j \in J} p_j^{b_j}, \quad b_j \geqslant 0.$$

Since f lies in $A[a^{-1}]$, $b_j \leqslant e_j$. For each j,

$$I[a^{-1}][a_j^{-1}] = I[(aa_j)^{-1}] = I[a_j^{-1}][a^{-1}].$$

Both g_j and g generate this ideal in $A[(aa_j)^{-1}]$. By unique factorization, $b_j = c_j$. This completes the proof. □

As a corollary, $\mathrm{Pic}(\mathbf{A}_R^n) = \{0\}$ for any UFD R.

Let us study the Picard group of projective space $\mathbf{P}_k^n$ over a field k. First, we construct some important invertible sheaves.

Construction. Consider the scheme $L = \{([x], t) \in \mathbf{P}_k^n \times_k \mathbf{A}_k^{n+1} : x_i t_j = t_i x_j\}$. While this point-set formula does not fully define a scheme, it captures the key idea. We can make this precise by writing down affine charts and using polynomial equations to define the scheme structure. This makes L a closed subscheme of $\mathbf{P}_k^n \times_k \mathbf{A}_k^{n+1}$.

The projection pr_1 restricts to a morphism $\pi\colon L \to \mathbf{P}_k^n$. As discussed earlier, this gives a line bundle over $\mathbf{P}_k^n$, called the ***tautological line bundle***. We denote its corresponding invertible sheaf by $\mathcal{O}_{\mathbf{P}_k^n}(-1)$ or simply $\mathcal{O}(-1)$. Its dual is denoted $\mathcal{O}_{\mathbf{P}_k^n}(1)$ or $\mathcal{O}(1)$. For any integer m, we define:

$$\mathcal{O}_{\mathbf{P}_k^n}(m) \text{ or simply } \mathcal{O}(m) := \begin{cases} \mathcal{O}(1)^{\otimes m} & m \geqslant 0 \\ \mathcal{O}(-1)^{\otimes(-m)} & m \leqslant 0. \end{cases}$$

The sheaf $\mathcal{O}(1)$ is called Serre's ***twisting sheaf***, and its corresponding line bundle is called the ***hyperplane bundle***.

Example 3. Let's compute the sections of $\mathcal{O}(m)$ over $\mathbf{P}_k^1$. First, recall that $\mathbf{P}_k^1$ is covered by two affine pieces: $U_0 = \mathrm{Spec}(k[T_1/T_0])$ and $U_1 = \mathrm{Spec}(k[T_0/T_1])$, glued along their intersection. We can view these rings as subrings of $k[T_0, T_1]$.

On U_0, the tautological bundle L trivializes as:

$$L \times_{\mathbf{P}_k^1} U_0 = \{([T_0, T_1], (t_0, t_1)) : T_0 \neq 0, T_0 t_1 = t_0 T_1\} \simeq \{(T_1/T_0, t_0)\} \simeq U_0 \times \mathbf{A}^1.$$

with similar trivialization on U_1:

$$L \times_{\mathbf{P}_k^1} U_1 = \{([T_0, T_1], (t_0, t_1)) : T_1 \neq 0, T_0 t_1 = t_0 T_1\} \simeq \{(T_0/T_1, t_1)\} \simeq U_1 \times \mathbf{A}^1.$$

On the overlap we have $(T_1/T_0, 1) \mapsto (T_0/T_1, T_1/T_0)$. So the transition function is given by $g_{01} = T_1/T_0$. Thus, for $\mathcal{O}(-1)$, a section would need functions $f(T_1/T_0)$ and $g(T_0/T_1)$ satisfying $T_1/T_0 \cdot f = g$. This is impossible, showing $\Gamma(\mathbf{P}_k^1; \mathcal{O}(-1)) = 0$.

For $\mathcal{O}(1)$, the transition function is $g_{01} = T_0/T_1$, sections should satisfy:

$$T_0/T_1 \cdot f(T_1/T_0) = g(T_0/T_1)$$

This means f and g come from a linear form $\ell(T_0, T_1) = aT_0 + bT_1$, with $f = T_0^{-1}\ell$ and $g = T_1^{-1}\ell$. Therefore:

$$\Gamma(\mathbf{P}^1_k; \mathcal{O}(1)) \simeq k^2$$

corresponds to degree 1 homogeneous polynomials in T_0, T_1.

More generally, for any m:

$$\Gamma(\mathbf{P}^1_k; \mathcal{O}(m)) = \begin{cases} 0 & m < 0 \\ \mathrm{Sym}^m(k^2) & m \geqslant 0 \end{cases}$$

The n-dimensional projective space $\mathbf{P}^n_k$ can be covered by $n+1$ affine spaces

$$U_i = \mathrm{Spec} k[T_0/T_i, \dots, T_n/T_i].$$

Using $g_{ij} = (T_i/T_j)^m$ as transition functions, we get $\mathcal{O}_{\mathbf{P}^n_k}(m)$. Similar to the computation above, we have

$$\Gamma(\mathbf{P}^n_k; \mathcal{O}(m)) = \begin{cases} 0 & m < 0 \\ \underbrace{\mathrm{Sym}^m(k^{n+1})}_{\text{homogeneous polynomials of degree } m} & m \geqslant 0 \end{cases}$$

Theorem. *The map*

$$m \mapsto \mathcal{O}(m) \colon \mathbf{Z} \to \mathrm{Pic}(\mathbf{P}^n_k)$$

is an isomorphism.

Proof. Since $\mathrm{Pic}(\mathbf{A}^n_k) = 0$, any invertible sheaf on $\mathbf{P}^n$ is trivial on $U_i = \{T_i \neq 0\} := \mathrm{Spec}(k[T_0/T_i, \dots, T_n/T_i])$. On $U_i \cap U_j = \mathrm{Spec}[T_0/T_i, \dots, T_n/T_i, T_i/T_j]$, one checks easily that

$$\mathcal{O}^*_X(U_i \cap U_j) = k^\times \cdot (T_i/T_j)^{\mathbf{Z}},$$

so $g_{ij} = u_{ij}(T_i/T_j)^{e_{ij}}$ for some $e_{ij} \in \mathbf{Z}$ and $u_{ij} \in k^\times$. The cocycle condition implies that

$$(T_i/T_j)^{e_{ij}}(T_j/T_k)^{e_{jk}} = (T_i/T_k)^{e_{ik}}, \quad \text{or equivalently } e_{ij} = e_{jk} = e_{ik}$$

and

$$u_{ij}u_{jk} = u_{ik}.$$

Thus we have $u_{ij} = u_{i1}/u_{j1}$ and there is e such that $e_{ij} = e$. So the transition functions g_{ij} is equivalent to $(T_i/T_j)^e$, which shows that the invertible sheaf is isomorphic to $\mathcal{O}(e)$. □

Theorem. *Let k be a field. Then* $\mathrm{Aut}_k(\mathbf{P}^n_k) \simeq \mathrm{PGL}_{n+1}(k)$.

Proof. Let φ be an automorphism of $\mathbf{P}^n_k$. Then $\varphi^*\mathcal{O}(1)$ must be a generator of $\mathrm{Pic}(\mathbf{P}^n_k)$ as well. There are only two such generators, and $\mathcal{O}(-1)$ has no nonzero section. So we must have $\varphi^*\mathcal{O}(1) = \mathcal{O}(1)$. But the space of sections of $\mathcal{O}(1)$ is identified with the set of linear forms in $n+1$ variables. So $\varphi^*(T_i) = \sum a_{ji}T_j$ for some invertible matrix $a_{ji} \in k$. This shows that φ is a projective transform. □